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Delay Gleichungen als abstrakte Differentialgleichungen in gewichteten Hilberträumen

Bachelorarbeit

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The $\LaTeX$-code can be found under https://gitlab.com/flukx/baarbeit.

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1. Introduction

1.1. Motivation

An analysis course for undergraduate students about ordinary differential equations (ODE) usually includes some solution techniques for special types of equations that can be solved explicitly together with the existence theorem of Peano and the existence and uniqueness theorem of Picard-Lindelőf. Both give local results and differentiable solutions. The right-hand side needs to be at least continuous.

One direction of research consists of *delay differential equations* (DDE). A DDE differs from an ODE by allowing a dependence of the right-hand side on the past of the unknown. One example from infection studies given in [Introduction HV93, p. 3] is

$$\dot{I}(t) = \lambda(I(t - L_1) - I(t - L_2)) \quad \text{for } t \in \mathbb{R}, I(t) \in \mathbb{R}_{>0}, L_1, L_2 \in \mathbb{R}_{<0}, \lambda \in \mathbb{R}_{>0}.$$

This is a DDE with discrete delay as studied in Section 7.3. This is not covered by the theorem of Picard-Lindelőf from usual undergraduate courses. There exist a big variety of solution techniques and solution property analysis, most of them limited to some classes of DDEs. For example [GP06] examines periodicity for equations with discrete delay with the background of mechanical applications and considers operators on function spaces of continuous function. Common topics are also stability and bifurcation, as in [Gop92]. A list of examples as well as a historical overview can be found in [HV93] which studies different types of DDEs like linear systems and equations of neutral type with a long list of tools.

For partial differential equations (PDEs), the approach of Picard-Lindelőf and explicit solution formulas do not suffice in most cases. Functional analysis and in particular Hilbert space theory on the other hand is a great tool to show the existence and uniqueness of solutions to many types of PDEs. Usually one gives up the requirement of differentiability of the solution and replace it with the notion of weak differentiability. A standard textbook on this is [Eva02].

The approach of [Kal+14] and my thesis uses suitable Hilbert spaces to formulate a solution theory for a wide class of ordinary differential equations. This includes the employment of weak differentiability. Then several types of delay differential equations are relatively simple special cases and the main motivation for the Hilbert space approach of the paper [Kal+14].

Applying the functional analysis notions of Hilbert space adjoints and weak differentiability to ordinary and delay differential equations might appear to be unnecessary complicated. Nevertheless, there are several reasons why it might be a reasonable alternative that can be taught in a seminar:

1. As just noted, being firm in Hilbert space theory is a strong background for PDEs afterwards. Especially, the concept of weak differentiability becomes familiar.

2. Once we can state Picard-Lindelőf using Hilbert space operators, the central contraction idea can stand out in its simplicity. In the classical proof of Picard-Lindelőf on the other hand, one has to take care of the domain of the involved Picard-iterates

and restrict it to a small neighborhood of the initial time. The used space of continuous functions with the supremum norm is a Banach space. The L_2-type spaces are Hilbert spaces, which enlarges the available toolbox.

3. The classical versions of Peano and Picard-Lindelőf only consider the case of differentiable solution and continuous right-hand sides. A wide range of applications is not covered by this though. Some examples are given in the introduction of [Háj79].

4. As mentioned above, classical solution theory is usually only local. In particular, it is necessary to restrict the solution to a small interval in order to make the Picard-iteration a contraction. Here, on the other hand, we will regard global solutions. That means, that we always regard functions that are defined on the entire real line. Instead of restricting the solution we choose the solution space appropriately.

   In [Section 5.2 Kal+14, 28 ff.] an application to local problems is given, but this will be beyond the scope of this paper.

With global in time solutions comes the notion of causality which captures the idea that a solution up to any time cannot depend on the future but only on the past and present. Hence, necessary properties of the differential equation that ensure causal solution operators, are presented. For a wider view on causality, see [Lak+10] which focusses solely on different types of causal differential equations.

1.2. Personal motivation

My aim is to give the reader a smooth guide to the outlined solution theory. This thesis is mostly self-contained, presenting just the parts and cases of Hilbert space operator theory needed.

My goal is to present not only statements and proofs but also ideas on why the regarded theory is developed and how one can go about proving the statements. I filled the gaps I found myself stuck in during studying the paper [Kal+14]. I hope that in this way others can easily follow the presented thoughts. To support this connection to [Kal+14], references to the respective sections are given throughout.

In order to limit the extend of this thesis, some topics of [Kal+14] are left out. They are not essential to the central solution theory.

I am grateful for Professor S. Siegmunds invitation to study this topic and the continued motivation as well as for the very supporting supervision by Dr. S. Trostorff. I also want to thank Niklas Jakob and Max Bender for their very detailed, valuable and constructive remarks.

1.3. Structure

Based on the work of the previous chapters, the solution theory is presented in Section 5. At first, the existence and uniqueness of solutions is discussed. Then in Section 6 the concept of causality is introduced.

A solution theory for (differential) equations has to analyse conditions on the equation that can guarantee the existence and uniqueness of a solution in a set of solution candidates. The candidates are a variant of L_2-spaces with an exponential weight and are introduced in Section 3. In order to allow distributions as right-hand sides, we embed the weighted L_2-spaces into so called extrapolation spaces. They are introduced in Section 4. On the other hand, this also allows us to get a glimpse on regularity theory in Section 5.2.

Since it is no more work to consider Hilbert space valued functions than $\mathbb{C}$ valued functions, we consider Hilbert space valued functions throughout. The background is presented in Section A.

Since L_2 includes non-differentiable functions, the question arises what differentiation as an operator on L_2 means. For this purpose we introduce differentiation as a closable operator on the Hilbert space $L_2(\mathbb{R})$. Hence the needed aspects on closures of (unbounded) linear operators are introduced in Section 2. Since differentiation is firstly defined on test functions, the density of test functions in $L_2(\mathbb{R})$ needs to be shown, see Section B. The extension to weighted L_2 spaces is done in Section 3.2 while the further extension to extrapolation spaces is done in Theorem 4.5.

After the general theory, some special cases that can be successfully tackled by the developed solution theory are inspected. This includes ordinary differential equations (Section 7.1), as well as delay differential equations with discrete (Section 7.3) and continuous delay (Section 7.4).

1.4. Notation

Throughout the thesis we use the following conventions.

All vector spaces are vector spaces over $\mathbb{C}$. The theory works the same over $\mathbb{R}$ though. By $L_2(\Omega, \Sigma, \mu; X)$ we denote the Hilbert space of square-integrable functions over the measure space (Ω, Σ, μ) mapping into the complex vector space X, factored out by equality almost everywhere. For the usual case of $\mathbb{R}$ with the Borel σ-algebra and the Lebesgue measure we write $L_2(\mathbb{R}; X)$ and omit the range of the functions in $L_2(\mathbb{R})$ in case of $X = \mathbb{C}$.

For any function f we write $\mathrm{D}(f)$ for the domain of f.

All operators on vector spaces are linear. For a vector space V we denote the vector space of all linear functionals into $\mathbb{C}$ by V'. For a Banach space X over $\mathbb{C}$ we denote the Banach space of all linear continuous functionals into $\mathbb{C}$ by X^*.

For a complex pre-Hilbert space H we denote the inner product of $f, g \in H$ by $\langle f, g \rangle_H$. If the space is clear from the context we may omit the index. The inner product is linear in the first argument and conjugate linear in the second argument.

For two sets A and B we write $A \subseteq B$ or $B \supseteq A$ if A is a subset of B or equal to B. We write $A \subset B$ or $B \supset A$ if A is a proper subset of B, i. e. not equal. In (almost) all cases of proper subsets in this thesis it is not of importance to exclude the equality case though.

The imaginary unit is i. For a complex number $z = x + \mathrm{i}y$ we denote the real part by $\operatorname{Re} z := x$ and the complex part by $\operatorname{Im} z := y$.

2. Closed and closable operators

This section summarizes some results from [IV.4, VII.2 Wer11, pp. 343 sqq.] that we will use to work with the differentiation operator ∂, which will be introduced later in Section 2.2.

Definition 2.1 (Closed operator, [IV.4.1 Wer11, p. 156])**.** Let X and Y be normed spaces, $D \subseteq X$ a subspace, $T\colon X \supseteq D \to Y$ linear. Then T is called *closed* if for every convergent sequence $(x_n)_{n\in\mathbb{N}}$, in D, $x_n \to x \in X$ with $Tx_n \to y \in Y$, we have $x \in D$ and $Tx = y$.

Note that this is weaker than continuity. Closedness of an operator T can also be viewed as closedness of its graph $\operatorname{gr}(T) = \{(x, Tx) \mid x \in D\} \subseteq X \times Y$ (under the norm $|||(x, y)||| = \|x\| + \|y\|$) as one can check. (See [IV.4.2 Wer11, pp. 156 sq.].)

Definition 2.2 (Closable operator, Closure, [VII.2.1 Wer11, p. 343])**.** An operator $T\colon X \supseteq \mathrm{D}(T) \to Y$ is called *closable* if there exists a closed extension B of T, that is $B\colon X \supseteq \mathrm{D}(B) \to Y$, such that B is closed and $\mathrm{D}(T) \subseteq \mathrm{D}(B)$ and $B|_{\mathrm{D}(T)} = T$. If B is the smallest closed extension, it is called the *closure* of T and $\operatorname{gr}(B) = \overline{\operatorname{gr}(T)}$ (closure in $X \times Y$). Hence we write $B = \overline{T}$ for the closure.

We write $T \subseteq S$ for two operators $T\colon X \supseteq \mathrm{D}(T) \to Y$, $S\colon X \supseteq \mathrm{D}(S) \to Y$ if $\operatorname{gr}(T) \subseteq \operatorname{gr}(S)$. That is, $\mathrm{D}(T) \subseteq \mathrm{D}(S)$ and $S|_{\mathrm{D}(T)} = T$.

From now on in this chapter we only consider operators on Hilbert spaces.

Closely connected to the concept of closed operators are adjoint operators. For continuous linear mappings $A\colon H \to H$ on Hilbert spaces there exist continuous adjoint operators A^* that satisfy

$$\langle Ax, y\rangle = \langle x, A^*y\rangle \text{ for all } x, y \in H.$$

In the case of (unbounded) operators the adjoint operator can still be defined but one has to be careful about the domain.

As a preparation we look at densely defined continuous operators first.

Lemma 2.3 (Continuous extension of continuous operators)**.** *Let X, Y be Hilbert spaces and $T\colon X \supseteq \mathrm{D}(T) \to Y$ be a densely defined, continuous operator. Then there is a unique extension of T to a continuous operator $\overline{T}: X \to Y$ (which is automatically the closure of T).*

Proof. In order to show that T can be continuously extended to X, we want to show that for all $\hat{x} \in X$ and all sequences $(x_i)_{i\in\mathbb{N}}$ in $\mathrm{D}(T)$ that converge to $\hat{x}$, the limit $\lim_{i\to\infty} Tx_i$ exists and is independent of the choice of the x_i. Then

$$\overline{T}\hat{x} := \lim_{i\to\infty} Tx_i$$

is well-defined and $\overline{T}$ is the only continuous extension of T.

Existence of the limit and continuity As $(x_i)_{i\in\mathbb{N}}$ is a Cauchy sequence and T is continuous, $(Tx_i)_{i\in\mathbb{N}}$ is a Cauchy sequence as well: $\|Tx_i - Tx_k\|_Y = \|T(x_i - x_k)\|_Y \leq \|T\|_{L(X,Y)} \|x_i - x_k\|$. Now, Y is a Hilbert space, so $(Tx_i)_{i\in\mathbb{N}}$ converges to some $\hat{y} \in Y$ and by the limit definition of $\overline{T}$, $\|y\|_Y = \|\overline{T}\hat{x}\|_Y \leq \|T\|_{L(X,Y)} \|\hat{x}\|_X$. That means, that $\overline{T}$ is continuous with the same operator norm as T.

Uniqueness Let $(x_i)_{i\in\mathbb{N}}$ and $(u_i)_{i\in\mathbb{N}}$ be two sequences in $\mathrm{D}(T)$, both converging to $\hat{x} \in X$, then $(x_i - u_i)_{i\in\mathbb{N}}$ converges to 0 and hence so does $(Tx_i - Tu_i)_{i\in\mathbb{N}} = (T(x_i - u_i))_{i\in\mathbb{N}}$, because T is continuous. This means $\lim_{i\to\infty} Tx_i = \lim_{i\to\infty} Tu_i$, so $\overline{T}$ is well-defined.

Linearity Let $\hat{x}, \hat{u} \in X$, $\lambda \in \mathbb{C}$ and and $(x_i)_{i\in\mathbb{N}}$ and $(u_i)_{i\in\mathbb{N}}$ be sequences in $\mathrm{D}(T)$ converging to $\hat{x}$ and $\hat{u}$, respectively. Then $x_i + \lambda u_i \to x + \lambda u$ and therefore

$$\begin{aligned}\overline{T}(\hat{x} + \lambda\hat{u}) &= \lim_{i\to\infty} T(x_i + \lambda u_i) = \lim_{i\to\infty} (Tx_i + \lambda T u_i) \\ &= \left(\lim_{i\to\infty} Tx_i\right) + \lambda \left(\lim_{i\to\infty} Tu_i\right) = T\hat{x} + \lambda T\hat{u}.\end{aligned}$$

□

Lemma 2.4 (Well-definedness of the adjoint operator, [VII.2 Wer11, p. 344])**.** *Let X, Y be Hilbert spaces and $T\colon X \supseteq \mathrm{D}(T) \to Y$ be a densely defined operator. Then consider*

$$\mathrm{D}(T^*) := \left\{ y \in Y \middle| x \mapsto \langle Tx, y\rangle_Y \text{ continuous on } \mathrm{D}(T) \right\}.$$

$\mathrm{D}(T^*)$ *is a linear subspace and for $y \in \mathrm{D}(T^*)$ we can extend $f\colon x \mapsto \langle Tx, y\rangle_Y$ to a unique continuous linear functional on X. Furthermore, there is a unique $z \in X$ such that $\langle Tx, y\rangle_Y = \langle x, z\rangle_X$ for all $x \in \mathrm{D}(T)$. We write $T^*y := z$. $T^*\colon Y \supseteq \mathrm{D}(T^*) \to X$ is a linear operator.*

Proof. Since $\langle\cdot,\cdot\rangle_Y$ is sesquilinear, $\mathrm{D}(T^*)$ is a linear subspace of Y. Let $y \in \mathrm{D}(T^*)$. Then the mapping $f\colon X \supseteq \mathrm{D}(T) \to \mathbb{C}$ defined by $f(x) = \langle Tx, y\rangle_Y$ is by definition densely defined, linear and continuous. By Lemma 2.3 f has a unique continuous extension to X.

By the Fréchet-Riesz representation theorem (see [Theorem V.3.6 Wer11, p. 228]) there is a unique $z \in X$ such that $\langle Tx, y\rangle_Y = \langle x, z\rangle_X$ for all $x \in \mathrm{D}(T)$.

The linearity of T^* follows from the uniqueness of z in the equation above, the sesquilinearity of the inner product and the linearity of T. □

Definition 2.5 (Adjoint operator, Self-adjoint, [VII.2.3 Wer11, p. 344])**.** The operator described in Lemma 2.4

$$T^*\colon Y \supseteq \mathrm{D}(T^*) \to X$$

with

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \text{for all } x \in \mathrm{D}(T), y \in \mathrm{D}(T^*)$$

is the *adjoint operator* of T.

If $T = T^*$ (which implies $\mathrm{D}(T) = \mathrm{D}(T^*)$ and $X = Y$), we call T *self-adjoint*. T is called *skew-self-adjoint* if $\mathrm{i}T$ is self-adjoint.

For a complex number $z \in \mathbb{C}$, z^* is the complex conjugate. This notation makes sense, since $\langle zx, y\rangle = \langle x, z^* y\rangle$ for x, y elements of a complex Hilbert space.

The adjoint operator has nice properties. First of all, it is closed:

Lemma 2.6 (Closedness of T^*, [VII.2.4 (a) Wer11, p. 345])**.** *Let X, Y be Hilbert spaces and $T\colon X \supseteq \mathrm{D}(T) \to Y$ be a densely defined operator. Then the adjoint operator T^* is closed.*

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{D}(T^*)$ with $y_n \to y \in Y$ and $T^* y_n \to \hat{x} \in X$ for $n \to \infty$. Then for $x \in \mathrm{D}(T)$

$$\langle Tx, y\rangle \overset{(*)}{=} \lim_{n\to\infty} \langle Tx, y_n\rangle = \lim_{n\to\infty} \langle x, T^* y_n\rangle$$
$$= \langle x, \hat{x}\rangle \overset{(*)}{\Longrightarrow} x \mapsto \langle Tx, y\rangle \text{ is continuous}$$

where in $(*)$ we have used the continuity of $\langle \cdot, \cdot\rangle$. This means $y \in \mathrm{D}(T^*)$ and $T^* y = \hat{x}$, as shown in Lemma 2.4. □

Secondly, the adjoint of an operator is also the adjoint of the closure:

Lemma 2.7. *Let X, Y be Hilbert spaces and $T\colon X \supseteq \mathrm{D}(T) \to Y$ be a densely defined closable operator. Then $T^* = \left(\overline{T}\right)^*$.*

Proof. Let $y \in \mathrm{D}\left(\left(\overline{T}\right)^*\right)$. Then $x \mapsto \left\langle \overline{T}x, y\right\rangle$ is continuous on $\mathrm{D}(\overline{T})$ and in particular on $\mathrm{D}(T) \subseteq \mathrm{D}(\overline{T})$. So $y \in \mathrm{D}(T^*)$ and since $\mathrm{D}(T)$ is dense in X, $T^* y = \left(\overline{T}\right)^* y$.

Let $y \in \mathrm{D}(T^*)$ and let $x \in \mathrm{D}(\overline{T})$. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathrm{D}(T)$ with $x_n \to x$ and $Tx_n \to Tx$. Hence

$$\left|\left\langle \overline{T}x, y\right\rangle\right| = \left|\lim_{n\to\infty} \langle Tx_n, y\rangle\right| = \left|\lim_{n\to\infty} \langle x_n, T^* y\rangle\right| \leq \lim_{n\to\infty} \|x_n\|_X \|T^* y\|_Y = \|x\|_X \|T^* y\|_Y.$$

So $x \mapsto \left\langle \overline{T}x, y\right\rangle$ is bounded by $\|T^* y\|_Y$ and hence $y \in \left(\overline{T}\right)^*$ and $T^* y = \left(\overline{T}\right)^* y$. □

Thirdly, the closure can be characterised by applying * twice:

Lemma 2.8 (Characterisation of closability [Theorem 1.8 Los13, p. 3])**.** *Let X, Y be Hilbert spaces and $T\colon X \supseteq \mathrm{D}(T) \to Y$ be a densely defined operator. Then $\mathrm{D}(T^*)$ is dense in Y if and only if T is closable. Furthermore, in this case $T^{**} = \overline{T}$ is the closure of T.*

Proof. For the easy direction assume that T^* has dense domain. Then T^{**} exists and for every $x \in \mathrm{D}(T)$, $y \in \mathrm{D}(T^*)$, we have $\langle x, T^* y\rangle = \langle Tx, y\rangle$, thus $y \mapsto \langle T^* y, x\rangle = \langle y, Tx\rangle$ is continuous which means that $x \in \mathrm{D}(T^{**})$ and $T^{**} x = Tx$.

So $T \subseteq T^{**}$ and T^{**} is closed by the Lemma 2.6, hence T^{**} is a closed extension of T.

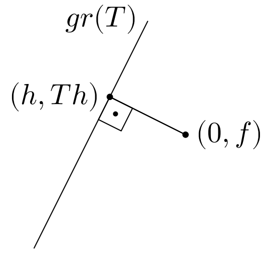

Figure 1: Projection theorem in $X \times Y$, see [Theorem 2.2 with Note 2.3 2 Alt12, pp. 314 sq.]

The other direction is more complicated. Assume that T is closable. Since $T^* = \left(\overline{T}\right)^*$ by Lemma 2.7 assume without loss of generality that T is closed. It is to be shown that $\mathrm{D}(T^*)$ is dense in Y. Take any $f \in \mathrm{D}(T^*)^\perp$. For this consider the minimization problem

$$M = \inf_{g \in \mathrm{D}(T)} \|f - Tg\|^2 + \|g\|^2 .$$

(Note that $M = 0$ if $f = 0$ which is what we want to show.)

The space $X \times Y$ with the inner product $\langle (g_1, f_1), (g_2, f_2) \rangle := \langle g_1, g_2 \rangle + \langle f_1, f_2 \rangle$ is an Hilbert space and $\mathrm{gr}(T)$ is a closed subspace. By the projection theorem illustrated in Figure 1 this infimum is attained by an $(h, Th) \in \mathrm{gr}(T)$ such that

$$\begin{gathered} M = \|(0, f) - (h, Th)\|^2 = \|h\|^2 + \|f - Th\|^2 \\ \text{and } \langle (v, Tv), (h, Th) - (0, f) \rangle_{X \times Y} = 0 \text{ for all } (v, Tv) \in \mathrm{gr}(T) \\ \text{which implies } \langle v, h \rangle_X = \langle Tv, f - Th \rangle_Y \text{ for all } v \in \mathrm{D}(T). \end{gathered}$$

This is the definition for $T^*(f - Th)$, that is

$$f - Th \in \mathrm{D}(T^*) \text{ and } T^*(f - Th) = h. \tag{1}$$

Since $f \perp \mathrm{D}(T^*)$, we can conclude with the Cauchy-Schwarz inequality

$$\langle f, f - Th \rangle = 0 \implies \|f\|^2 = \langle f, f \rangle = \langle f, Th \rangle \leq \|f\| \, \|Th\| \implies \|f\| \leq \|Th\| . \tag{2}$$

Further

$$\begin{aligned} &\|h\|^2 = \langle h, h \rangle \overset{(1)}{=} \langle h, T^*(f - Th) \rangle = \langle Th, f - Th \rangle = \langle Th, f \rangle - \|Th\|^2 = \|f\|^2 - \|Th\|^2 \\ &\implies \|h\|^2 + \|Th\|^2 = \|f\|^2 \overset{(2)}{\leq} \|Th\|^2 \\ &\implies \|h\|^2 \leq 0 \implies h = 0 \implies Th = 0 \\ &\overset{(1)}{\implies} f - Th = f \in \mathrm{D}(T^*) \cap \mathrm{D}(T^*)^\perp \implies f = 0. \end{aligned}$$

This shows that $\mathrm{D}(T^*)$ is dense in Y.

Now we have $\overline{T} \subseteq T^{**}$ and want to show $T^{**} \subseteq \overline{T}$ which means by definition that $\mathrm{gr}(T^{**}) \subseteq \overline{\mathrm{gr}(T)} \subseteq X \times Y$ as mentioned in Definition 2.2.

As before we use the inner product $\langle (u,v),(x,y)\rangle_{X\times Y} := \langle u,x\rangle_X + \langle v,y\rangle_Y$ on $X \times Y$. For $\overline{\mathrm{gr}(T)} \supseteq \mathrm{gr}(T^{**})$ it is enough to show that $\mathrm{gr}(T)^\perp \subseteq \mathrm{gr}(T^{**})^\perp$. So let $(u,v) \in \mathrm{gr}(T)^\perp$, i. e. $\langle x,u\rangle_X + \langle Tx,v\rangle_Y = 0$ for all $x \in \mathrm{D}(T)$. Then $x \mapsto \langle Tx,v\rangle = \langle x,-u\rangle$ is continuous, hence $v \in \mathrm{D}(T^*)$ and $T^*v = -u$. For $(z, T^{**}z) \in \mathrm{gr}(T^{**})$ we then have

$$\begin{aligned}\langle (z,T^{**}z),(u,v)\rangle_{X\times Y} &= \langle z,u\rangle_X + \langle T^{**}z,v\rangle_Y = \langle z,u\rangle_X + \langle z,T^*v\rangle_X \\ &= \langle z,u+T^*v\rangle_X = \langle z,0\rangle_X = 0 \\ &\implies (u,v) \perp \mathrm{gr}(T^{**})^\perp\end{aligned}$$

This is what we wanted to show. □

In the following we consider operators from one Hilbert space to itself and therefore call this Hilbert space H.

An important class of operators are the symmetric ones:

Definition 2.9 (Symmetric operator, [VII.2.2 Wer11, p. 344])**.** Let $T\colon H \supset \mathrm{D}(T) \to H$ be a densely defined operator with $\langle Tx,y\rangle = \langle x,Ty\rangle$ for all $x,y \in \mathrm{D}(T)$. Then T is called *symmetric*.

This definition of symmetry appears to be almost the same as self-adjoint but the domains (for $T = T^*$, $\mathrm{D}(T) = \mathrm{D}(T^*)$ is required) are different. That is why those two terms should not be confused. (Also see [Ch. 9 Hal13, pp. 169 sqq.].)

Lemma 2.10 (Symmetric operators are closable, [Proposition 9.4 Hal13, p. 171])**.** *Let $T\colon H \supset \mathrm{D}(T) \to H$ be a symmetric operator. Then T is closable with $\overline{T} \subseteq T^*$.*

Proof. Let $y \in \mathrm{D}(T)$. Then $x \mapsto \langle Tx,y\rangle = \langle x,Ty\rangle$ is clearly continuous, since $\langle\cdot,\cdot\rangle$ is continuous and for all $x \in \mathrm{D}(T)$, $\langle x,Ty\rangle = \langle Tx,y\rangle = \langle x,T^*y\rangle$ holds. Hence $T \subseteq T^*$. In every metric space we have for two sets $A \subseteq B \implies \overline{A} \subseteq \overline{B}$, so $\overline{\mathrm{gr}(T)} \subseteq \overline{\mathrm{gr}(T^*)} \overset{2.6}{=} \mathrm{gr}(\overline{T^*}) = \mathrm{gr}(T^*)$ and with Definition 2.2 this is $\overline{T} \subseteq T^*$. □

2.1. Spectrum of self-adjoint operators

In order to solve differential equations one essential tool is integration, namely the inverse of differentiation. It is well-known that an anti-derivative is not unique. Speaking in terms of operators, differentiation does not have an inverse. To circumvent this issue we will use a slightly modified version of the differential operator, which is invertible (see Corollary 3.5). Which type of “a little” is possible can be asked in a much more general case where we arrive at the notion of spectra. It is the generalisation of eigenvalues in the infinite dimensional setting.

Definition 2.11 (Spectrum)**.** Let $T\colon X \supseteq \mathrm{D}(T) \to X$ be a densely defined operator on a Banach space. Then

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is not continuously invertible}\}$$

is called the *spectrum of* T. Continuously invertible includes that the domain of the inverse is all of X.

Remark. In the finite dimensional case a linear operator is invertible if and only if it is injective. Hence in this case the spectrum is the set of eigenvalues.

Lemma 2.12 (Real spectrum of self-adjoint operators, [VII.2.16 Wer11, pp. 355 sq.])**.** *The spectrum of a self-adjoint operator* T *on a Hilbert space* H *is contained in the real axis. For any* $\lambda \in \mathbb{C} \setminus \mathbb{R}$ *we have the bound*

$$\left\|(T-\lambda)^{-1}\right\| \leq \operatorname{Im}(\lambda) \tag{3}$$

for the operator norm of the inverse of $T - \lambda$.

Proof. Let $T\colon H \supseteq \mathrm{D}(T) \to H$ be a self-adjoint operator operator on a Hilbert space H. This implies T is densely defined and closed.

Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. It is to be shown that $T - \lambda$ is bounded below (implies existence and boundedness of the inverse) and has dense range (implies together with boundedness that $\mathrm{D}((T-\lambda)^{-1}) = H$).

Let $x \in \mathrm{D}(T)$ with $\|x\| = 1$. In order to find a lower bound for $\|(T-\lambda)x\|$ it is unpractical that T and λ appear on both sides in the inner product $\|(T-\lambda)x\|^2 = \langle (T-\lambda)x, (T-\lambda)x\rangle$. That is why one starts with the Cauchy Schwarz inequality:

$$\|(T-\lambda)x\| = \|(T-\lambda)x\|\,\|x\| \geq |\langle (T-\lambda)x, x\rangle| \overset{*}{=} |\langle Tx, x\rangle - \lambda| \overset{**}{\geq} |\operatorname{Im}(\lambda)| > 0.$$

Here $*$ uses $1 = \|x\| = \|x\|^2 = \langle x, x\rangle$ and $**$ uses that T is self-adjoint by noting

$$\langle x, Tx\rangle^* = \langle Tx, x\rangle = \langle x, Tx\rangle \implies \langle Tx, x\rangle \in \mathbb{R}.$$

This shows a lower bound for $\|(T-\lambda)x\|$ independent of x and hence $T - \lambda$ is injective with bounded inverse $(T-\lambda)^{-1}\colon H \supseteq (T-\lambda)(\mathrm{D}(T)) \to H$. (3) follows.

To show that the domain of the inverse is H use the trick to consider the orthogonal complement: Let $z \in H$ with $\langle (T-\lambda)x, z\rangle = 0$ for all $x \in \mathrm{D}(T)$. Then

$$\begin{aligned} 0 = \langle (T-\lambda)x, z\rangle &= \langle Tx, z\rangle - \langle x, \lambda^* z\rangle && \text{for all } x \text{ in } \mathrm{D}(T) \\ \implies \langle x, \lambda^* z\rangle &= \langle Tx, z\rangle && \text{for all } x \text{ in } \mathrm{D}(T) \\ \implies z \in \mathrm{D}(T^*) &= \mathrm{D}(T) \text{ and } (T^* - \lambda^*)z = (T - \lambda^*)z = 0 \end{aligned}$$

Since $T - \lambda^*$ is injective by the first part of the proof, we have $z = 0$. So $(T-\lambda)(\mathrm{D}(T))^\perp = \{0\}$ and therefore $T - \lambda$ has dense range.

Since $(T-\lambda)^{-1}$ is continuous $\mathrm{D}((T-\lambda)^{-1}) = (T-\lambda)^{-1}(H)$ is closed and dense in H and therefore equal to H. ☐

2.2. Closure of the time derivative

In this section the time derivative ∂_c on $C_c^\infty(\mathbb{R})$ and the closure ∂ on a larger domain will be discussed. One main class of functions that is discussed in every section of analysis are differentiable functions. Here differentiation is introduced as a linear operator, firstly only on smooth functions with compact support $C_c^\infty(\mathbb{R})$. As we will see in the following, this operator can be closed and in this way define a notion of weak differentiability.

Definition 2.13 ([Definition 2.1 Kal+14, p. 7])**.** Let

$$\begin{aligned} \partial_c \colon L_2(\mathbb{R}) \supset C_c^\infty(\mathbb{R}) &\to L_2(\mathbb{R}) \\ \phi &\mapsto \phi' \\ \text{and define } \partial &:= -{\partial_c}^*. \end{aligned}$$

Then ∂_c is skew-symmetric, ∂ is well-defined and $\overline{\partial_c} \subseteq \partial$ by Lemma 2.10. We call $f \in \mathrm{D}(\partial)$ *weakly differentiable.*

Proof. The skew-symmetry of ∂_c is just integration by parts. Let $\psi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned} \psi \mapsto \langle \partial_c \psi, \psi \rangle &= \int_{\mathbb{R}} \psi'(x)\psi(x)^* \, dx \; = [\psi(x)\psi(x)^*]_{-\infty}^{+\infty} - \int_{\mathbb{R}} \psi(x)\psi'(x)^* \, dx \\ &= - \int_{\mathbb{R}} \psi(x)\psi'(x)^* \, dx \; = - \langle \psi, \psi' \rangle \\ &= \langle \psi, -\partial_c \psi \rangle \text{ is continuous on } C_c^\infty(\mathbb{R}). \end{aligned}$$

$C_c^\infty(\mathbb{R})$ is dense in $L_2(\mathbb{R})$ by Theorem B.10. From Lemma 2.10 we get that $\mathrm{i}\partial_c \subset (\mathrm{i}\partial_c)^*$ but this implies directly $\partial_c \subset (-\mathrm{i})(\mathrm{i}^*){\partial_c}^* = -{\partial_c}^* = \partial$ and the desired statement. ☐

One would naturally expect that the closure of a symmetric operator is self-adjoint but this is indeed not the case in general. One example is given by differentiation on absolutely continuous functions, see [Example (a) after Definition VII.2.6 Wer11, pp. 347 sq.] for details. In our case of differentiation on test functions as a subspace of $L_2(\mathbb{R})$ it is the case though, but the argument takes several steps. For ∂ being skew-self-adjoint, we have to show $\partial = -\partial^* = -(-{\partial_c}^*)^* \overset{2.8}{=} \overline{\partial_c}$. That is, for any $u \in \mathrm{D}(\partial)$ we have to find a sequence $(u_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R})$ such that $u_n \to u$ and $\partial_c u_n \to \partial u$ in $L_2(\mathbb{R})$.

In Section B we have seen how to approximate functions in $L_2(\mathbb{R})$ with test functions: truncate and convolute. Since in the first step of truncation we want to stay in $\mathrm{D}(\partial)$ we cannot truncate with $\chi_{[-n,n]}$. Instead use

$$\begin{aligned} (\eta_n)_{n\in\mathbb{N}} &\text{ in } C_c^\infty(\mathbb{R}) \text{ such that} \\ \operatorname{supp}(\eta_n) &\subset [-n, n], \\ \eta_n(x) &\leq 1 \text{ for all } x \in \mathbb{R}, \\ \eta_n(x) &= 1 \text{ for all } x \in [-n+1, n-1], \\ \eta_n|_{[-n,-n+1]} &= \eta_m|_{[-m,-m+1]}(\cdot + n - m) \text{ for all } n, m > 1, \\ \text{and } \; \eta_n|_{[n,n-1]} &= \eta_m|_{[m,m-1]}(\cdot - n + m) \text{ for all } n, m > 1, \\ \text{Then } \operatorname{supp}(\eta_n') &\subset [-n, -n+1] \cup [n-1, n]. \end{aligned} \tag{4}$$

This is the formalisation of a bump that becomes more and more stretched towards positive and negative infinity as n grows. In particular η_n' consists of the same two bumps that move out towards positive and negative infinity as n grows and hence the η_n's are uniformly bounded.

Together with the convolution we define with δ_m being a Friedrichs mollifier as in Definition B.2

$$u_{n,m} := \delta_m * (\eta_n u). \tag{5}$$

In order to relate $\partial u_{n,m}$ to ∂u we have to check how ∂ interact with convolution and multiplication. For both we have statements for smooth functions (product rule and Theorem B.6) which we want to generalise.

Lemma 2.14 (Product rule)**.** *Let $\varphi \in C_c^\infty(\mathbb{R})$ and $v \in \mathrm{D}(\partial)$. Then $\varphi v \in \mathrm{D}(\partial)$ and*

$$\partial(\varphi v) = (\partial\varphi)v + \varphi(\partial v) = \varphi' v + \varphi(\partial v).$$

Proof. Since ∂ is defined as an adjoint operator we show the equation by testing it with any $\psi \in C_c^\infty(\mathbb{R})$:

$$\begin{aligned}
\langle \partial\varphi v + \varphi\partial v, \psi\rangle_{L_2(\mathbb{R})} &= \int_{\mathbb{R}} \varphi' v\psi^* + \varphi\partial v\psi^* = \langle v, \varphi'^*\psi\rangle_{L_2(\mathbb{R})} + \langle \partial v, \varphi^*\psi\rangle_{L_2(\mathbb{R})} \\
&= \langle v, \varphi'^*\psi\rangle_{L_2(\mathbb{R})} - \langle v, \partial(\varphi^*\psi)\rangle_{L_2(\mathbb{R})} \\
&= \langle v, \varphi'^*\psi\rangle_{L_2(\mathbb{R})} - \langle v, \varphi'^*\psi + \varphi^*\psi'\rangle_{L_2(\mathbb{R})} \\
&= -\langle v, \varphi^*\psi'\rangle_{L_2(\mathbb{R})} = -\int_{\mathbb{R}} v\varphi\psi'^* = -\langle \varphi v, \psi'\rangle_{L_2(\mathbb{R})}.
\end{aligned}$$

Since v, ∂v are in $L_2(\mathbb{R})$ and φ and $\partial\varphi$ are bounded, $\partial\varphi v + \varphi\partial v$ is in $L_2(\mathbb{R})$ and hence $\psi \mapsto \langle \varphi v, \psi'\rangle_{L_2(\mathbb{R})}$ is continuous on $C_c^\infty(\mathbb{R})$. By definition of $\partial = -{\partial_c}^*$, $\varphi v \in \mathrm{D}(\partial)$ and $\partial(\varphi v) = \partial\varphi v + \varphi\partial v$. □

Lemma 2.15 (Differentiation of convolution)**.** *Let $\varphi \in C_c^\infty(\mathbb{R})$ and $u \in \mathrm{D}(\partial)$. Then $\varphi * u \in \mathrm{D}(\partial)$ and*

$$\partial(\varphi * u) = (\partial\varphi) * u = \varphi * (\partial u).$$

Proof. $(\partial\varphi) * u$ is in $L_2(\mathbb{R})$ since $\partial\varphi \in C_c^\infty(\mathbb{R})$ and $u \in L_2(\mathbb{R})$ and by B.6 we have $(\partial\varphi) * u = \partial(\varphi * u)$. To show the last equality we again test with $\psi \in C_c^\infty(\mathbb{R})$. The use of the theorem of Fubini is justified since the last term is finite and all integrands are non-negative. Note that we use that (classical) differentiation and argument shift commutate

$$
\begin{aligned}
-\langle \varphi * u, \partial\psi\rangle_{L_2(\mathbb{R})} &= -\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(t)u(x-t)\,dt\,\psi'(x)^*\,dx \\
&= -\int_{\mathbb{R}} \varphi(t)\int_{\mathbb{R}} u(x-t)\psi'(x)^*\,dx\;dt && \text{(Fubini)} \\
&= -\int_{\mathbb{R}} \varphi(t)\int_{\mathbb{R}} u(y)\psi'(y+t)^*\,dy\;dt && (y = x-t) \\
&= \int_{\mathbb{R}} \varphi(t)\int_{\mathbb{R}} \partial u(y)\psi(y+t)^*\,dy\;dt && \text{(Definition } \partial \text{ for } u \text{ and } \psi(\cdot+t)) \\
&= \int_{\mathbb{R}} \varphi(t)\int_{\mathbb{R}} \partial u(x-t)\psi(x)^*\,dy\;dt && (x = y+t) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(t)\partial u(x-t)\,dt\,\psi(x)^*\,dx && \text{(Fubini)} \\
&= \int_{\mathbb{R}} (\varphi * \partial u)\psi^*\,dx \\
&= \langle \varphi * \partial u, \psi\rangle_{L_2(\mathbb{R})}
\end{aligned}
$$

So, $\partial(\varphi * u) = \varphi * (\partial u)$. □

Theorem 2.16 ($\partial = \overline{\partial_c}$, [Theorem 2.2 Kal+14, p. 7])**.** *∂ is skew-self-adjoint, that is $\partial = \overline{\partial_c}$ by the argument on page 13.*

Proof. Let $u \in \mathrm{D}(\partial)$ and $u_{m,n}$ as in (5). Then $u_{m,n} \in C_c^\infty(\mathbb{R})$ for all $m, n \in \mathbb{N}$ and

$$\partial u_{m,n} = \partial(\delta_m * (\eta_n u)) \overset{2.15}{=} \delta_m * \partial(\eta_n u) \overset{2.14}{=} \delta_m * ((\partial\eta_n)u + \eta_n(\partial u)).$$

Let $\varepsilon > 0$. Since $\operatorname{supp}\partial\eta_n \subset [-n, -n+1] \cup [n-1, n]$ and $\partial\eta_n$ are uniformly bounded, and hence $\partial\eta_n u \to 0$ for $n \to \infty$, there is $N_1 \in \mathbb{N}$, such that for all $n > N_1$, $\|\partial\eta_n u\|_{L_2(\mathbb{R})} < \frac{\varepsilon}{4}$.

By the dominated convergence theorem (dominated by ∂u) there is $N_2 \in \mathbb{N}$ such that $\|\eta_n\partial u - \partial u\|_{L_2(\mathbb{R})} < \frac{\varepsilon}{4}$.

By Theorem B.9 there exists for every $n \in \mathbb{N}$ an $m_n \in \mathbb{N}$ such that

$$\left\|\delta_{m_n} * (\partial\eta_n u + \eta_n\partial u) - (\partial\eta_n u + \eta_n\partial u)\right\|_{L_2(\mathbb{R})} < \frac{\varepsilon}{2}.$$

With those estimates and the triangle inequality we get

$$\left\|\delta_{m_n} * (\partial\eta_n u + \eta_n\partial u) - \partial u\right\|_{L_2(\mathbb{R})} < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$$

for all $n > \max\{N_1, N_2\}$. Hence $\partial u_{m_n,n} \to \partial u$ and $u_{m_n,n} \to u$ for $n \to \infty$ in $L_2(\mathbb{R})$. That is by definition that $u \in \mathrm{D}(\overline{\partial_c})$ and $\overline{\partial_c}u = \partial u$. □

3. Weighted function spaces

As it was mentioned in the introduction (Section 1.3) we want to work with Hilbert space valued functions, not only $\mathbb{C}$-valued ones. Fortunately it does not hurt to always think of $\mathbb{C}$-valued functions since the theory does not change thanks to the results of Section A.

Let H be any Hilbert space throughout the remaining thesis.

3.1. Antiderivatives of test functions

In order to write a differential equation as a fixed point problem as we will do in Section 5 we need to study the inverse of the differentiation operator ∂.

Since ∂ is the usual derivative on test functions $\varphi \in C_c^\infty(\mathbb{R}; H)$, the inverse $\partial^{-1}\varphi$ must be an antiderivative, given by $\int_{x_0}^x \varphi(t)\,dt$ for some $x_0 \in \mathbb{R}$. Since constant functions apart from 0 are not in $L_2(\mathbb{R}; H)$, at most one antiderivative Φ is in $L_2(\mathbb{R}; H)$.

Φ must be constant for sufficiently small and large $(\mathbb{R} \setminus \operatorname{supp}(\varphi))$ arguments, call these values $\Phi(-\infty)$ and $\Phi(\infty)$. They are related via $\Phi(\infty) = \Phi(-\infty) + \int_{\mathbb{R}} \varphi$ by the fundamental theorem of calculus.

Unfortunately this implies that Φ is not in $L_2(\mathbb{R}; H)$ except for the special case $\int_{\mathbb{R}} \varphi = 0$. Hence we need a different space that contains functions that are constant from some point onward. Since ∂ needs to be defined for those functions as well we can inductively conclude that one of the following two function spaces must be included in our setting:

Definition 3.1 (Test space, [Definition 3.1 Kal+14, p. 12])**.** Define

$$C_{c+}^\infty(\mathbb{R}; H) := \left\{ \phi \in C^\infty(\mathbb{R}; H) \,\middle|\, \begin{array}{l} \sup \operatorname{supp} \phi < \infty \\ \text{and there is } n \in \mathbb{N} \text{ with } \phi^{(n)} \in C_c^\infty(\mathbb{R}; H) \end{array} \right\}$$

and

$$C_{c-}^\infty(\mathbb{R}; H) := \left\{ \phi \in C^\infty(\mathbb{R}; H) \,\middle|\, \begin{array}{l} \inf \operatorname{supp} \phi > -\infty \\ \text{and there is } n \in \mathbb{N} \text{ with } \phi^{(n)} \in C_c^\infty(\mathbb{R}; H) \end{array} \right\}$$

(The $+/-$ in the notation should hint to the fact, that the compact support only applies to the positive/ negative side of the real line.)

3.2. Weighted L_2-space

In order to apply methods of functional analysis we want to work in a Hilbert space similar to $L_2(\mathbb{R}; H)$. In order to include $C_{c\pm}^\infty(\mathbb{R})$ we introduce an exponential weight.

Definition 3.2 (weighted Space, [Kal+14, Def. 2.3])**.** Let $\varrho \in \mathbb{R}$. Define

$$H_\varrho^0(\mathbb{R}) := \left\{ f \in L_{2,\mathrm{loc}}(\mathbb{R}) \,\middle|\, (x \mapsto \exp(-\varrho x) f(x)) \in L_2(\mathbb{R}) \right\}$$
$$H_\varrho^0(\mathbb{R}) \otimes H = \left\{ f \in L_{2,\mathrm{loc}}(\mathbb{R}) \otimes H \,\middle|\, (x \mapsto \exp(-\varrho x) f(x)) \in L_2(\mathbb{R}) \otimes H \right\}$$

We endow $H^0_\varrho(\mathbb{R})$ and $H^0_\varrho(\mathbb{R}) \otimes H$ respectively with the inner products

$$(f,g) \mapsto \langle f,g\rangle_{\varrho,0} := \int_{\mathbb{R}} f(x)g(x)^* \exp(-2\varrho x)\,dx$$
$$(f,g) \mapsto \langle f,g\rangle_{\varrho,0} = \int_{\mathbb{R}} \langle f(x),g(x)\rangle_H \exp(-2\varrho x)\,dx$$

and the induced norms, both called $\|\cdot\|_{\varrho,0}$. Here $L_{2,\mathrm{loc}}(\mathbb{R})$ is the space of all locally square-integrable functions

$$L_{2,\mathrm{loc}}(\mathbb{R}) = \left\{ f\colon \mathbb{R} \to \mathbb{C} \,\middle|\, f \text{ measurable}, \forall K \subset \mathbb{R} \text{ compact } \int_K \|f(x)\|^2\,dx < \infty \right\}$$

(of course factored out by equality almost everywhere).

Note that $H^0_0(\mathbb{R}) = L_2(\mathbb{R})$, $H^0_0(\mathbb{R}) \otimes H = L_2(\mathbb{R}) \otimes H$ and $\|\cdot\|_{0,0} = \|\cdot\|_{L_2(\mathbb{R})}$.

Remark (Unitary $\exp(-\varrho m)$, [after Definition 2.3 Kal+14, p. 8]). $H^0_\varrho(\mathbb{R}) \otimes H$ is obviously isometrically isomorphic to $L_2(\mathbb{R}; H)$ with the following unitary operator $\exp(-\varrho m)$:

$$\exp(-\varrho m)\colon H^0_\varrho(\mathbb{R}) \otimes H \to L_2(\mathbb{R}) \otimes H\colon f \mapsto (x \mapsto \exp(-\varrho x) f(x)), \tag{6}$$

with m being multiplication with the argument:

$$(mf)(x) = xf(x) \quad (x \in \mathbb{R}).$$

Here exp is *not* to be understood pointwise:

$$\exp(-\varrho m)(f)(x) \neq \exp(-\varrho(mf)(x)) = \exp(-\varrho x f(x)).$$

Instead remember the definition of exp on $\mathbb{R}$: $\exp(x) = \sum_{k=0}^\infty \frac{x^k}{k!}$. Here exp is applied to the linear operator $-\varrho m$ and the multiplication used to define the powers of the argument is concatenation. For $f \in H^0_\varrho(\mathbb{R})$, $x \in \mathbb{R}$

$$\begin{aligned}
\exp(-\varrho m) &= \sum_{k=0}^\infty \frac{(-\varrho m)^k}{k!} \\
\implies \exp(-\varrho m)(f) &= \sum_{k=0}^\infty \frac{1}{k!}(-\varrho m)^k(f) \\
\implies \exp(-\varrho m)(f)(x) &= \sum_{k=0}^\infty \frac{1}{k!}(-\varrho m)^k(f)(x) = \sum_{k=0}^\infty \frac{1}{k!}(-\varrho)^k x^k f(x) \\
&= \left(\sum_{k=0}^\infty \frac{1}{k!}(-\varrho)^k x^k \right) f(x) = \exp(-\varrho x) f(x).
\end{aligned}$$

This is also the canonical generalisation of exp used on matrices where concatenation and (matrix) multiplication coincide.

On $L_2(\mathbb{R})$ (and $L_2(\mathbb{R}) \otimes H$) we developed the differentiation as a linear operator with the greatest possible domain in Section 2.2. In order to expand this operator to the weighted spaces we can use the unitary operators $\exp(-\varrho m)$ to go from $H^0_\varrho(\mathbb{R}) \otimes H$ to $L_2(\mathbb{R}) \otimes H$, differentiate there and go back:

Definition 3.3 (Naive differentiation on weigted space, [Corollary 2.5 Kal+14, p. 8])**.** For $\varrho \in \mathbb{R}$ define the (unbounded) linear operator $\tilde{\partial}_\varrho$ on

$$\begin{aligned} \mathrm{D}(\tilde{\partial}_\varrho) &= \exp(-\varrho m)^{-1}(\mathrm{D}(\partial)) \subset H^0_\varrho(\mathbb{R}) \otimes H \\ \tilde{\partial}_\varrho &\colon H^0_\varrho(\mathbb{R}) \otimes H \supset \mathrm{D}(\tilde{\partial}_\varrho) \to H^0_\varrho(\mathbb{R}) \otimes H \\ \tilde{\partial}_\varrho &:= \exp(-\varrho m)^{-1} \partial \exp(-\varrho m). \end{aligned} \tag{7}$$

After defining $\tilde{\partial}_\varrho$ one can ask if that is what we would expect of the derivative in the cases that we can calculate directly: $C_c^\infty(\mathbb{R})$. Let $\varphi \in C_c^\infty(\mathbb{R}) \subseteq H^0_\varrho(\mathbb{R})$. Then

$$\begin{aligned} (\tilde{\partial}_\varrho \varphi)(x) &= \exp(-\varrho m)^{-1} \partial (x \mapsto \exp(-\varrho x)\varphi(x)) \\ &= \exp(-\varrho m)^{-1} (x \mapsto -\varrho \exp(-\varrho x)\varphi(x) + \exp(-\varrho x)\varphi'(x)) \\ &= \exp(-\varrho m)^{-1} \exp(-\varrho m)(x \mapsto -\varrho\varphi(x) + \varphi'(x)) \\ &= -\varrho\varphi + \varphi' \end{aligned} \tag{8}$$

This is not exactly what we desired since generalised differentiation should be the classical derivative on smooth functions. That is why we introduce a correction term.

Definition 3.4 (Differentiation on weigted space, [Corollary 2.5 Kal+14, p. 8])**.** For $\varrho \in \mathbb{R}$ define

$$\partial_\varrho := \tilde{\partial}_\varrho + \varrho. \tag{9}$$

The calculation in (8) also verifies that ∂_ϱ does not depend on ϱ except for the domain. (Also see [after Corollary 2.5 Kal+14, p. 9].)

So far all we achieved is an expansion of our space of functions but the actual gain is that differentiation becomes an invertible operator on $H^0_\varrho(\mathbb{R}) \otimes H$ for $\varrho \neq 0$. Here we use Lemma 2.12: $\varrho \in \mathbb{R}$ and therefore ϱ is not in the spectrum of $\tilde{\partial}_\varrho$.

Corollary 3.5 (∂_ϱ continuously invertible)**.** *For $\varrho \in \mathbb{R} \setminus \{0\}$ the previously in 3.4 defined operator ∂_ϱ has a continuous inverse with*

$$\|\partial_\varrho^{-1}\|_{L(H^0_\varrho(\mathbb{R})\otimes H, H^0_\varrho(\mathbb{R})\otimes H)} = \frac{1}{|\varrho|}.$$

Proof. By Lemma 2.16 $\mathrm{i}\partial$ is self-adjoint and hence by Lemma 2.12 the spectrum of $\mathrm{i}\partial$ is contained in the real axis. This means by definition, that $\mathrm{i}\partial - \mathrm{i}\varrho$ is continuously invertible for all $\varrho \in \mathbb{R} \setminus \{0\}$ and Lemma 2.12 gives the estimate

$$\|(\partial - \varrho)^{-1}\|_{L(L_2(\mathbb{R}), L_2(\mathbb{R}))} = \|\mathrm{i}(\partial - \varrho)^{-1}\|_{L(L_2(\mathbb{R}), L_2(\mathbb{R}))} \geq \frac{1}{|\varrho|}.$$

In order to show $\left\|\mathrm{i}(\partial-\varrho)^{-1}\right\|_{L(L_2(\mathbb{R}),L_2(\mathbb{R}))} \leq \frac{1}{|\varrho|}$ we could find an element $f \in L_2(\mathbb{R})$ with $\|f\|_{0,0}$ such that $\left\|(\partial-\varrho)^{-1}f\right\|_{0,0} = \frac{1}{|\varrho|}$. This exists if we find a $g \in L_2(\mathbb{R})\setminus\{0\}$ with $\partial g = 0$ since with appropriate scaling we can define $f = (\partial-\varrho)g$ and get

$$1 = \|f\|_{0,0} = \|(\partial-\varrho)g\|_{0,0} = \|-\varrho g\|_{0,0} = |\varrho| \left\|(\partial-\varrho)^{-1}f\right\|_{0,0}$$

The typical candidates would be constant functions but those are not in $L_2(\mathbb{R})$, so we need to find an approximating sequence. Define

$$g_n := \frac{\delta * \chi_{[-n,n]}}{\left\|\delta * \chi_{[-n,n]}\right\|_{0,0}}$$

with $\delta = \delta_1$ being a mollifier as defined in Definition B.2. Then $g_n \in C_c^\infty(\mathbb{R})$ and hence we can calculate

$$\|\partial(g_n)\|_{0,0} = \|g_n'\|_{0,0} \overset{B.6}{=} \left\| \frac{\delta' * \chi_{[-n,n]}}{\left\|\delta * \chi_{[-n,n]}\right\|_{L_2(\mathbb{R})}} \right\|_{0,0} = \underbrace{\left\|\delta' * \chi_{[-n,n]}\right\|_{0,0}}_{\text{constant},n\to\infty} \underbrace{\left\|\delta * \chi_{[-n,n]}\right\|_{0,0}^{-1}}_{\to 0,\, n\to\infty} \to 0$$

δ' is a function with support in $[-1,1]$ and integral 0. Hence $\delta' * \chi_{[-n,n]}$ consists of two bumps on $[-n-1,-n+1]$ and $[n-1,n+1]$ that move to $\pm\infty$ for $n\to\infty$ but look the same for all n. Hence $\left\|\delta' * \chi_{[-n,n]}\right\|_{0,0}$ is constant over all $n \in \mathbb{N}$.

This tells us

$$\|(\partial-\varrho)g_n\|_{L_2(\mathbb{R})} \leq \|\partial g_n\|_{0,0} + |\varrho| \, \|g_n\|_{0,0} \to |\varrho| \, \|g_n\|_{0,0}\,,$$
$$\text{hence } \left\|(\partial-\varrho)^{-1}\right\|_{L(L_2(\mathbb{R}),L_2(\mathbb{R}))} \leq \frac{1}{|\varrho|}.$$

Since $\exp(-\varrho m)$ is unitary, $\left\|\partial_\varrho^{-1}\right\|_{L(\varrho,0,\varrho,0)} = \left\|(\partial+\varrho)^{-1}\right\|_{L(L_2(\mathbb{R}),L_2(\mathbb{R}))} = \frac{1}{|\varrho|}$.

∂_ϱ^{-1} on $H_\varrho^0(\mathbb{R}) \otimes H$ has the same norm as discussed in example A.8. □

Now that we have properly introduced the inverse of differentiation we can summarize the conclusions of Section 3.1:

Corollary 3.6 (Explicit formula for antiderivatives, [Corollary 2.5 (d) Kal+14, p. 8])**.** *Let $\varrho \in \mathbb{R}_{>0}$, $f \in H_\varrho^0(\mathbb{R}) \otimes H$. Then*

$$\left(\partial_\varrho^{-1} f\right)(x) = \int_{-\infty}^{x} f(t)\,dt\,. \tag{10}$$

Let $f \in H_{-\varrho}^0(\mathbb{R}) \otimes H$. Then

$$\left(\partial_{-\varrho}^{-1} f\right)(x) = \int_{\infty}^{x} f(t)\,dt = -\int_{x}^{\infty} f(t)\,dt \tag{11}$$

Proof. As we have seen in Section 3.1 the result holds for $f \in C_c^\infty(\mathbb{R}; H)$. By continuity we want to extend this result to all of $H_\varrho^0(\mathbb{R}) \otimes H$. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $C_c^\infty(\mathbb{R}; H)$ converging to f in $H_\varrho^0(\mathbb{R}) \otimes H$. By continuity proven in Corollary 3.5 $\partial_\varrho^{-1}\varphi_n$ converges to $\partial_\varrho^{-1} f$. Now consider $\Phi_n = \partial_\varrho^{-1}\varphi_n = \int_{-\infty}^x \varphi(t)\, dt$ for $n \in \mathbb{N}$ and $F = \int_{-\infty}^x f(t)\, dt$ and estimate

$$\begin{aligned}
\|\Phi_n - F\|_{\varrho,0} &= \int_{\mathbb{R}} \left| \int_{-\infty}^x \varphi_n(t)\, dt - \int_{-\infty}^x f(t)\, dt \right|^2 \exp(-2\varrho x)\, dx \\
&= \int_{\mathbb{R}} \left| \int_{-\infty}^x \varphi_n(t) - f(t)\, dt \right|^2 \exp(-2\varrho x)\, dx \\
&\le \int_{\mathbb{R}} \int_{-\infty}^x |\varphi_n(t) - f(t)|^2 \ dt \ \exp(-2\varrho x)\, dx && \text{(Jensen-inequality)} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{[-\infty,x]}(t) \, |\varphi_n(t) - f(t)|^2 \exp(-2\varrho x)\, dt \ dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{[t,\infty]}(x) \, |\varphi_n(t) - f(t)|^2 \exp(-2\varrho x)\, dt \ dx \\
&= \int_{\mathbb{R}} \int_t^\infty \exp(-2\varrho x)\, dx \ |\varphi_n(t) - f(t)|^2 \ dt && \text{(Fubini)} \\
&= \int_{\mathbb{R}} \left[\frac{-1}{2\varrho} \exp(-2\varrho x) \right]_t^\infty |\varphi_n(t) - f(t)|^2 \ dt \\
&= \int_{\mathbb{R}} \frac{1}{2\varrho} \exp(-2\varrho t) \, |\varphi_n(t) - f(t)|^2 \ dt \\
&= \frac{1}{2\varrho} \|\varphi_n - f\|_{\varrho,0} \to 0, \quad n \to \infty.
\end{aligned}$$

This shows $\partial_\varrho^{-1} f \leftarrow \partial_\varrho^{-1}\varphi_n = \Phi_n \to F$ in $H_\varrho^0(\mathbb{R}) \otimes H$, so $F = \partial_\varrho^{-1} f$.

The second case for $H_{-\varrho}^0(\mathbb{R}) \otimes H$ is proven analogously. □

4. Gelfand triple

So far we have defined the differentiation on a sub (vector) space $\mathrm{D}(\partial_\varrho)$ of $H^0_\varrho(\mathbb{R}) \otimes H$ and as ∂_ϱ is unbounded there is no way to extend it continuously onto all of $H^0_\varrho(\mathbb{R}) \otimes H$ mapping to $H^0_\varrho(\mathbb{R}) \otimes H$. In order to apply methods from functional analysis we rather want to consider an operator on a Hilbert space, not on a non-complete pre-Hilbert space. So we make $\mathrm{D}(\partial_\varrho)$ a Hilbert space in the following way:

Definition 4.1 ($H^1_\varrho(\mathbb{R})$, [Def. 2.6 Kal+14, p. 9])**.** For $\varrho \in \mathbb{R} \setminus \{0\}$ define $H^1_\varrho(\mathbb{R})$ as $(\mathrm{D}(\partial_\varrho), \langle\cdot,\cdot\rangle_{\varrho,1})$ with the inner product $\langle f, g\rangle_{\varrho,1} = \left\langle \partial_\varrho f, \partial_\varrho g\right\rangle_{\varrho,0}$ for $f, g \in H^1_\varrho(\mathbb{R})$. It follows for the norm: $\|f\|_{\varrho,1} = \left\|\partial_\varrho f\right\|_{\varrho,0}$.

Remark. Note that even though $\mathrm{D}(\partial_\varrho)$ is as a vector space a subspace of $H^0_\varrho(\mathbb{R}) \otimes H$, $H^1_\varrho(\mathbb{R}) \otimes H$ is *not* a sub Hilbert space because $\|\cdot\|_{\varrho,1} \neq \|\cdot\|_{\varrho,0}$.

Equipped with $H^1_\varrho(\mathbb{R}) \otimes H$ we could write our equation to hold in $H^0_\varrho(\mathbb{R}) \otimes H$ with the solution being an element of $H^1_\varrho(\mathbb{R}) \otimes H$ but there are common use cases where the right hand side of a differential equation cannot be formulated as a function but as a linear functional on functions. The most prominent example are initial value problems as described in [Section 5.1 Kal+14, p. 24]. We model this by the following space.

Definition 4.2 (Extrapolation space $H^{-1}_\varrho(\mathbb{R})$)**.** For $\varrho \in \mathbb{R}\setminus\{0\}$ define $H^{-1}_\varrho(\mathbb{R}) := H^1_{-\varrho}(\mathbb{R})^*$ as the space of all continuous linear functionals on $H^1_{-\varrho}(\mathbb{R})$ with the usual operator norm, denoted by $\|\cdot\|_{\varrho,-1}$. Pay attention to the sign of ϱ! $Hi\varrho{-}1$ can be called *extrapolation space.*

$H^{-1}_\varrho(\mathbb{R}) \otimes H \cong \left(H^1_{-\varrho}(\mathbb{R}) \otimes H\right)^*$ as expected.

Remark (Degree of differentiability). The index -1, 0 and 1 indicate the degree of differentiability. Functions in $H^1_\varrho(\mathbb{R}) \otimes H$ can be differentiated once in the sense of ∂_ϱ. Functions in $H^0_\varrho(\mathbb{R}) \otimes H$ in general cannot be derived but we will see in Theorem 4.5 that we can make sense of ∂_ϱ as an operator from $H^0_\varrho(\mathbb{R}) \otimes H$ to $H^{-1}_\varrho(\mathbb{R}) \otimes H$, so "first antiderivatives" of elements of $H^{-1}_\varrho(\mathbb{R}) \otimes H$ are not differentiable but second antiderivatives are, justifying the intuition of a differentiability degree of -1.

Since ∂_ϱ^{-1} is continuous as a mapping from $H^0_\varrho(\mathbb{R}) \otimes H$ to $H^0_\varrho(\mathbb{R}) \otimes H$, the identity map is a continuous embedding of $H^1_\varrho(\mathbb{R}) \otimes H$ into $H^0_\varrho(\mathbb{R}) \otimes H$. We want to have the same for $H^0_\varrho(\mathbb{R}) \otimes H$ into $H^{-1}_\varrho(\mathbb{R}) \otimes H$ and get it by a typical identification of functions with functionals: integration. We want to identify a function u with the functional

$$\psi \mapsto \int_{\mathbb{R}} \langle u, \psi\rangle_H = \langle u, \psi\rangle_{0,0}.$$

Since neither elements of $H^0_\varrho(\mathbb{R}) \otimes H$ nor elements of $H^1_\varrho(\mathbb{R}) \otimes H$ are square-integrable one needs $\exp(-\varrho m)$. That would yield for $f \in H^0_\varrho(\mathbb{R}) \otimes H$, $g \in H^1_\varrho(\mathbb{R}) \otimes H$:

$$\underbrace{\langle f, g\rangle_{0,0}}_{\text{not well-defined}} = \underbrace{\int_{\mathbb{R}} \langle f, g\rangle_H}_{\text{We want that}} \neq \underbrace{\langle f, g\rangle_{\varrho,0}}_{\text{defined}}$$

The trick chosen here is to not use $g \in H^1_\varrho(\mathbb{R}) \otimes H$ but $g \in H^1_{-\varrho}(\mathbb{R}) \otimes H$. This way the exp cancel and the inner product in $L_2(\mathbb{R}) \otimes H$ is well-defined while calculated in the naive way. This is the reason, why $H^{-1}_\varrho(\mathbb{R}) \otimes H$ is defined as $\left(H^1_{-\varrho}(\mathbb{R}) \otimes H\right)^*$ and not $\left(H^1_\varrho(\mathbb{R}) \otimes H\right)^*$.

$$\underbrace{\langle f, g\rangle_{0,0}}_{\text{not well-defined}} = \underbrace{\int_{\mathbb{R}} \langle f, g\rangle_H}_{\text{We want that}} = \underbrace{\int_{\mathbb{R}} \left\langle \overbrace{\exp(-\varrho m)f}^{\in L_2(\mathbb{R})\otimes H}, \overbrace{\exp(\varrho m)g}^{\in L_2(\mathbb{R})\otimes H} \right\rangle}_{\text{We do have this}} =: \underbrace{\langle f, g\rangle_{0,0}}_{\text{Now defined!}}$$

Lemma 4.3 (Embedding, [Remark 2.7 Kal+14, p. 9])**.** *A function ϕ in $H^0_\varrho(\mathbb{R}) \otimes H$ is identified with the following linear continuous functional in $H^{-1}_\varrho(\mathbb{R}) \otimes H$:*

$$\psi \mapsto \langle \exp(-\varrho m)\phi, \exp(\varrho m)\psi\rangle_{0,0} =: \langle \phi, \psi\rangle_{0,0} \tag{12}$$

This embedding is conjugate-linear and continuous with

$$\|\psi\|_{\varrho,1} \leq \frac{1}{\varrho} \|\psi\|_{\varrho,0}. \tag{13}$$

In the same way we identify $H^0_\varrho(\mathbb{R}) \otimes H$ with $(H^0_{-\varrho}(\mathbb{R}) \otimes H)^*$. By the Fréchet-Riesz representation theorem (see [Theorem V.3.6 Wer11, p. 228]) and since $\exp(\pm\varrho m)$ *is isometric this identification is isometric.*

Proof. Call the embedding ι. It is to be checked that for $f \in H^0_\varrho(\mathbb{R}) \otimes H$ the functional $\iota(f)$ is indeed linear, continuous and the ι is conjugate-linear and continuous. Let $\phi \in H^0_\varrho(\mathbb{R}) \otimes H$, $\psi \in H^1_{-\varrho}(\mathbb{R}) \otimes H$. The linearity of $\iota(\phi)$ and the conjugate-linearity of ι are clear since $\exp(\pm\varrho m)$ is linear and $\langle \cdot, \cdot\rangle$ is conjugate bilinear. Then we have to find a $C \in \mathbb{R}$ independent of ψ ($\implies$ the functional is continuous) and ϕ ($\implies$ the embedding is continuous) such that

$$\left|\langle \exp(-\varrho m)\phi, \exp(\varrho m)\psi\rangle_{0,0}\right| \leq C \|\psi\|_{-\varrho,1} \|\phi\|_{\varrho,0}$$

The typical tool for estimation of inner products is the Cauchy-Schwarz inequality. The other ingredient is the fact that the identity embedding from $H^1_{-\varrho}(\mathbb{R})\otimes H$ into $H^0_{-\varrho}(\mathbb{R})\otimes H$ is continuous because $\partial^{-1}_{-\varrho}$ is continuous:

$$\begin{aligned}
\left|\langle \exp(-\varrho m)\phi, \exp(\varrho m)\psi\rangle_{0,0}\right| &\leq \|\exp(-\varrho m)\phi\|_{0,0} \|\exp(\varrho m)\psi\|_{0,0} \\
&= \|\phi\|_{\varrho,0} \|\psi\|_{-\varrho,0} \\
&= \|\phi\|_{\varrho,0} \left\|\partial^{-1}_{-\varrho}\partial_{-\varrho}\psi\right\|_{-\varrho,0} \\
&\qquad \text{(well-defined since } \psi \in \mathrm{D}(\partial_{-\varrho})) \\
&\leq \|\phi\|_{\varrho,0} \left\|\partial^{-1}_{-\varrho}\right\|_{L(H^0_{-\varrho}(\mathbb{R}), H^0_{-\varrho}(\mathbb{R}))} \left\|\partial_{-\varrho}\psi\right\|_{-\varrho,0} \\
&\overset{\text{Def. 4.1}}{=} \|\phi\|_{\varrho,0} \left\|\partial^{-1}_{-\varrho}\right\|_{L(H^0_{-\varrho}(\mathbb{R}), H^0_{-\varrho}(\mathbb{R}))} \|\psi\|_{-\varrho,1}
\end{aligned}$$

$$\overset{3.5}{=} \|\phi\|_{\varrho,0} \frac{1}{|-\varrho|} \|\psi\|_{-\varrho,1}$$

$$\implies \|\phi\|_{\varrho,-1} \leq \frac{1}{|\varrho|} \|\phi\|_{\varrho,0}$$

With $C = \frac{1}{|\varrho|} = \frac{1}{|-\varrho|}$ we are done.

□

At this point it also becomes evident why we have to choose $\varrho \neq 0$. Otherwise the identity is not a continuous embedding from $H^1_{-\varrho}(\mathbb{R})$ into $H^0_{-\varrho}(\mathbb{R})$ and this construction does not work.

Remark (Gelfand triple, [Theorem 2.8 Kal+14, p. 9]). The triple

$$(H^1_{-\varrho}(\mathbb{R}) \otimes H, H^0_0(\mathbb{R}) \otimes H, H^{-1}_{\varrho}(\mathbb{R}) \otimes H)$$

with $H^0_0(\mathbb{R}) \otimes H = L_2(\mathbb{R}) \otimes H$ identified with $H^0_{-\varrho}(\mathbb{R}) \otimes H$ and $H^0_{\varrho}(\mathbb{R}) \otimes H$ via $\exp(\pm\varrho m)$ is called a *Gelfand triple*.

Definition 4.4 (Rigged Hilbert space). In general a Gelfand triple or *rigged Hilbert space* is a triple (K_0, K_1, K_0^*) with K_1 being a Hilbert space, $K_0 \hookrightarrow K_1$ a densely embedded vector space with a finer topology such that the inclusion map $\iota\colon K_0 \to K_1$ is continuous. Then ι^* maps from K_0^* to $K_1^* \cong K_1$, is continuous as well with $\iota^*\varphi(\psi) = \langle\varphi, \psi\rangle_{K_1}$ for $\varphi \in K_0$, $\psi \in K_1$ and embeds K_1 densely into K_0^*.

Proof. For simplicity assume that K_0 is a Hilbert space as well. In the case we are interested in this is the case and the proof for the general case needs no additional idea.

First consider the operator norm of ι^*. Let $\varphi \in K_1$, $\psi \in K_0$ with $\|\psi\|_{K_0} = 1$. Then

$$\begin{aligned} |(\iota^*\varphi)(\psi)| = \left|\langle\varphi, \iota\psi\rangle_{K_1}\right| &\leq \|\varphi\|_{K_1} \|\iota\|_{L(K_0,K_1)} \|\psi\|_{K_0} \\ \implies \|\iota^*\varphi\|_{K_0^*} &\leq \|\varphi\|_{K_1} \|\iota\|_{L(K_0,K_1)} \\ \implies \|\iota^*\|_{L(K_1,K_0^*)} &\leq \|\iota\|_{L(K_0,K_1)} . \end{aligned}$$

So ι^* is continuous. ι^* is injective since $\iota(K_0)$ is dense in K_1 and $\langle\varphi_1, \psi\rangle_{K_1} = \langle\varphi_2, \psi\rangle_{K_1}$ for $\varphi_1, \varphi_2 \in K_1$ and all ψ in a dense subset of K_1 implies $\varphi_1 = \varphi_2$.

We want to show that ι^* has dense range, that is $(\iota^*)^{-1}$ has dense domain. First show that taking the adjoint and the inverse commute. Note that $(\iota^{-1})^*$ is well-defined since ι^{-1} is densely defined. Then we have the equivalences

$$\begin{aligned} \Phi \in \mathrm{D}\left((\iota^*)^{-1}\right) &\iff \exists\varphi \in K_1 \forall\psi \in K_0 : \Phi(\psi) = \langle\varphi, \iota\psi\rangle_{K_1} \\ &\iff \exists\varphi \in K_1 \forall\varsigma \in \iota(K_0) : \Phi\left(\iota^{-1}\varsigma\right) = \langle\varphi, \varsigma\rangle_{K_1} \\ &\iff \left(\varsigma \mapsto \Phi\left(\iota^{-1}\varsigma\right)\right) \text{ is continuous on } \iota\left(K_0\right) \iff \Phi \in \mathrm{D}\left((\iota^{-1})^*\right) . \end{aligned}$$

Here we see that $\mathrm{D}\left((\iota^{-1})^*\right) = \mathrm{D}\left((\iota^*)^{-1}\right)$ and in both cases the image of Φ is φ.

By Lemma 2.8 $(\iota^{-1})^*$ is densely defined if and only if ι^{-1} is closable. But $\operatorname{gr}\left(\iota^{-1}\right)$ is the same as $\operatorname{gr}(\iota)$ except that the entries got swapped. Since ι is continuous $\operatorname{gr}(\iota) \subset K_0 \times K_1$ is closed and therefore ι^{-1} is closed as well. Hence $\left(\iota^{-1}\right)^* = (\iota^*)^{-1}$ is densely defined. That means that ι^* has dense range. $\square$

Gelfand triples are in general studied to consider generalised eigenvalues of operators, most prominently differentiation as in our case. An introduction to the motivation and use of Rigged Hilbert spaces in physics can be found in [Mad01].

4.1. Dualities

We have seen in Lemma 2.16 that ∂ is skew-self-adjoint which means that we can easily move differentiation from one side to the other in inner products.

Since that is sometimes useful in calculations we check how this duality transfers to ∂_ϱ. (Also see [Remark 2.9 (a), (b) Kal+14, p. 10].)

For $\phi \in H^1_\varrho(\mathbb{R}) \otimes H$, $\psi \in H^1_{-\varrho}(\mathbb{R}) \otimes H$ we have

$$\begin{aligned}\left\langle \partial_\varrho \phi, \psi \right\rangle_{0,0} &\overset{4.3}{=} \left\langle \exp(-\varrho m) \exp(-\varrho m)^{-1} (\partial + \varrho) \exp(-\varrho m)\phi, \exp(\varrho m)\psi \right\rangle_{0,0} \\ &\overset{\text{Def. 3.4}}{=} \left\langle \exp(-\varrho m)\phi, (-\partial + \varrho) \exp(\varrho m)\psi \right\rangle_{0,0} \\ &= \left\langle \exp(-\varrho m)\phi, -\exp(\varrho m) \exp(\varrho m)^{-1} (\partial - \varrho) \exp(\varrho m)\psi \right\rangle_{0,0} \\ &\overset{\text{Def. 3.4}}{=} \left\langle \phi, -\partial_{-\varrho}\psi \right\rangle_{0,0}.\end{aligned} \tag{14}$$

For the inverse ∂_ϱ^{-1} this immediately yields for $\varrho \neq 0$ and $\phi \in H^0_\varrho(\mathbb{R}) \otimes H$ and $\psi \in H^0_{-\varrho}(\mathbb{R}) \otimes H$ by plugging in $\partial_\varrho^{-1}\phi \in H^1_\varrho(\mathbb{R}) \otimes H$ and $\partial_{-\varrho}^{-1}\psi \in H^1_{-\varrho}(\mathbb{R}) \otimes H$ into (14):

$$\left\langle \partial_\varrho^{-1}\phi, \psi \right\rangle_{0,0} = \left\langle \phi, \partial_{-\varrho}^{-1}\psi \right\rangle_{0,0} \tag{15}$$

Theorem 4.5 (Derivation as an unitary operator, [Theorem 2.8 Kal+14, p. 9])**.** *Let $\varrho \in \mathbb{R} \setminus \{0\}$. Then the following mappings are unitary:*

$$\begin{aligned}\partial_{-\varrho} := \partial_{1\to 0} \colon H^1_{-\varrho}(\mathbb{R}) \otimes H &\to H^0_{-\varrho}(\mathbb{R}) \otimes H \\ \phi &\mapsto \partial_{-\varrho}\phi \\ \partial_{-\varrho}^{-1} := \partial_{0\to 1} \colon H^0_{-\varrho}(\mathbb{R}) \otimes H &\to H^1_{-\varrho}(\mathbb{R}) \otimes H \\ \phi &\mapsto \partial_{-\varrho}^{-1}\phi \\ \partial_\varrho := \partial_{0\to -1} \colon H^0_\varrho(\mathbb{R}) \otimes H &\to H^{-1}_\varrho(\mathbb{R}) \otimes H \\ \phi &\mapsto \left(H^1_{-\varrho}(\mathbb{R}) \otimes H \ni \psi \mapsto \left\langle -\partial_{-\varrho}\psi, \phi \right\rangle_{0,0} \right) \\ \partial_\varrho^{-1} := \partial_{-1\to 0} \colon H^{-1}_\varrho(\mathbb{R}) \otimes H &\to H^0_\varrho(\mathbb{R}) \otimes H \\ \phi &\mapsto \left(H^0_{-\varrho}(\mathbb{R}) \otimes H \ni \psi \mapsto \phi(-\partial_{-\varrho}^{-1}\psi) \right)\end{aligned}$$

where the last definition is to be understood similarly to the identification defined in 4.3.

Note that the symbols $\partial_\varrho^{\pm 1}$ are used for different operators that only differ in the range and domain. Hence while reading one always has to take care to distinguish which domain ∂_ϱ is operating on.

Proof. The first two mappings $\partial_{1\to 0}$ and $\partial_{0\to 1}$ are unitary by definition of $H^1_{-\varrho}(\mathbb{R}) \otimes H$.

If one wants to define $\partial_\varrho (= \partial_{0\to -1})$ on $H^0_\varrho(\mathbb{R}) \otimes H$ this new definition should coincide with the old one on $\mathrm{D}(\partial_\varrho)$. With (14) this holds for the definition of $\partial_{0\to-1}$ for $\phi \in \mathrm{D}(\partial_\varrho) \subset H^0_\varrho(\mathbb{R}) \otimes H$. Fortunately the term $\left\langle \phi, \partial_{-\varrho}\psi \right\rangle_{0,0}$ can also be defined for any $\phi \in H^0_\varrho(\mathbb{R}) \otimes H$, giving motivation to the definition of $\partial_{0\to-1}$. To show that it is indeed isometric and therefore unitary verify for $\phi \in H^0_\varrho(\mathbb{R}) \otimes H$ with $\|\phi\|_{\varrho,0} = 1$:

$$\begin{aligned}
\|\partial_{0\to-1}\phi\|_{\varrho,-1} &= \sup_{\substack{\psi\in H^1_{-\varrho}(\mathbb{R})\otimes H\\ \|\psi\|_{-\varrho,1}=1}} \left|\left\langle \phi, -\partial_{-\varrho}\psi\right\rangle_{0,0}\right| \\
&= \sup_{\substack{\psi\in H^1_{-\varrho}(\mathbb{R})\otimes H\\ \|\psi\|_{-\varrho,1}=1}} \left|\left\langle \exp(-\varrho m)\phi, -\exp(\varrho m)\partial_{-\varrho}\psi\right\rangle_{0,0}\right| \\
&\le \sup_{\substack{\psi\in H^1_{-\varrho}(\mathbb{R})\otimes H\\ \|\psi\|_{-\varrho,1}=1}} \|\exp(-\varrho m)\phi\|_{0,0} \left\|-\exp(\varrho m)\partial_{-\varrho}\psi\right\|_{0,0} \quad \text{(Cauchy-Schwarz)} \\
&= \sup_{\substack{\psi\in H^1_{-\varrho}(\mathbb{R})\otimes H\\ \|\psi\|_{-\varrho,1}=1}} \|\phi\|_{\varrho,0} \|\psi\|_{-\varrho,1} = 1
\end{aligned}$$

$$\begin{aligned}
\text{and } \|\partial_{0\to-1}\phi\|_{\varrho,-1} &\ge \left|\left\langle \exp(-\varrho m)\phi, -\exp(\varrho m)\partial_{-\varrho} \underbrace{\left(-\partial_{-\varrho}^{-1}\exp(-2\varrho m)\phi\right)}_{\in H^1_{-\varrho}(\mathbb{R})\otimes H} \right\rangle_{0,0}\right| \\
&= \left|\langle \exp(-\varrho m)\phi, \exp(-\varrho m)\phi\rangle_{0,0}\right| = \|\phi\|^2_{\varrho,0} = 1
\end{aligned}$$

since $1 = \|\phi\|_{\varrho,0} = \left\|-\partial_{-\varrho}^{-1}\exp(-2\varrho m)\phi\right\|_{-\varrho,1}$

In order to verify that $\partial_{-1\to0}$ is unitary we show that $\partial_{-1\to0} = \partial_{0\to-1}^{-1}$. Let $\phi \in H^0_\varrho(\mathbb{R}) \otimes H$. Then

$$\begin{aligned}
\partial_{-1\to0}\partial_{0\to-1}\phi &= \partial_{-1\to0}\left(\psi \mapsto \left\langle -\partial_{-\varrho}\psi, \phi\right\rangle\right) \\
&= \left(\psi \mapsto \left\langle -\partial_{-\varrho}\left(-\partial_{-\varrho}^{-1}\psi\right), \phi\right\rangle_{0,0}\right) \\
&= \left(\psi \mapsto \langle \psi, \phi\rangle_{0,0}\right) = \phi \qquad \text{via the embedding of 4.3}
\end{aligned}$$

Since $-\partial_{-\varrho}^{-1}$ is isometric $\partial_{-1\to0}\phi = 0$ implies $\phi = 0$. Hence $\partial_{-1\to0}$ is injective, and together with $\partial_{-1\to0}\partial_{0\to-1} = \mathrm{id}_{H^0_\varrho(\mathbb{R})\otimes H}$ it is bijective. □

5. Solution Theory

After introducing differentiation as an operator between different function spaces we now use this setting to tackle (ordinary) differential equations. Usually when a differential equation arises in any application it is not given which properties a solution must have, i. e. in which space a solution is to be found. This gives the opportunity to choose a function space in which a solution is searched for.

An ordinary differential equation can be written in the form

$$x'(t) = f(x(t), t) \quad t \in \mathbb{R}$$

with $f\colon H \times \mathbb{R} \to \mathbb{R}$. In order to simplify and to consider the whole function x as once and not pointwise, we write

$$\partial_\varrho x = F(x) \text{ (here) with } F\colon x \mapsto (t \mapsto f(x(t), t)). \tag{16}$$

Additionally this generalization allows even more cases as we will see later in the applications. The main idea is to write this equivalently as

$$x = \partial_\varrho^{-1} F(x)$$

and use the contraction mapping theorem. For this $\partial_\varrho^{-1} F$ must be a contraction on a suitable space.

The biggest space that we have seen so far on which we were able to define differentiation is $H_\varrho^0(\mathbb{R}) \otimes H$ with $\partial_\varrho\colon H_\varrho^0(\mathbb{R}) \otimes H \to H_\varrho^{-1}(\mathbb{R}) \otimes H$. Hence we have to view F as a function from $H_\varrho^0(\mathbb{R}) \otimes H$ to $H_\varrho^{-1}(\mathbb{R}) \otimes H$.

Via the embedding of Lemma 4.3 it is possible to extend F if it is defined on smooth functions and maps to $C_c^\infty(\mathbb{R}; H) \subset H_\varrho^0(\mathbb{R}) \otimes H$ but in many applications this is not suitable. One example is given in [Introduction and Section 5.1 Kal+14, pp. 5, 24]. There an initial value problem is formulated in terms of linear functionals on a test space.

We we have constructed the test space $C_{c+}^\infty(\mathbb{R}; H)$ in Definition 3.1 such that integration is always possible and it lies in $H_{-\varrho}^1(\mathbb{R}) \otimes H$. Hence F must be given as a mapping from $C_c^\infty(\mathbb{R}; H)$ to $C_{c+}^\infty(\mathbb{R}; H)'$

5.1. Picard-Lindelőf

In order to consider an element of $C_{c+}^\infty(\mathbb{R}; H)'$ as an element of $H_\varrho^{-1}(\mathbb{R}) \otimes H$ it must be continuous with respect to $\|\cdot\|_{-\varrho,1}$. Hence we need the following condition on $F\colon C_c^\infty(\mathbb{R}; H) \to C_{c+}^\infty(\mathbb{R}; H)'$: For all $u \in C_c^\infty(\mathbb{R}; H)$, there exists $K \in \mathbb{R}$, such that for all $\psi \in C_{c+}^\infty(\mathbb{R}; H)$ we have:

$$|F(u)(\psi)| \leq K \|\psi\|_{-\varrho,1}. \tag{17}$$

As noted above $\partial_\varrho^{-1} F$ must be a contraction in order to use the contraction mapping theorem. Since ∂_ϱ^{-1} is unitary from $H_\varrho^{-1}(\mathbb{R}) \otimes H$ to $H_\varrho^0(\mathbb{R}) \otimes H$, F must be a contraction

(i. e. Lipschitz continuous with Lipschitz constant less than 1) itself. Let the Lipschitz constant of F be called s. Then we need the following condition for all $u, w \in C_c^\infty(\mathbb{R}; H)$:

$$\|F(u) - F(w)\|_{\varrho,-1} \leq s \|u - w\|_{\varrho,0} .$$

F is not mapping to $H_\varrho^{-1}(\mathbb{R}) \otimes H$ by definition. So we write this without the $H_\varrho^{-1}(\mathbb{R}) \otimes H$-norm: (Also see [equation (4), 2nd part Kal+14, p. 13].)

$$|F(u)(\psi) - F(w)(\psi)| \leq s \|\psi\|_{-\varrho,1} \|u - w\|_{\varrho,0} \text{ for all } \psi \in C_{c+}^\infty(\mathbb{R}; H). \tag{18}$$

Now we can simplify the first condition (17) to

$$\text{There exists } K \in \mathbb{R} \text{ such that for all } \psi \in C_{c+}^\infty(\mathbb{R}; H)\text{: } |F(0)(\psi)| \leq K \|\psi\|_{-\varrho,1} \tag{19}$$

since (19) together with (18) implies for any $\phi \in C_c^\infty(\mathbb{R}; H)$ and any $\psi \in C_{c+}^\infty(\mathbb{R}; H)$

$$\begin{aligned} |F(\phi)(\psi)| &= |(F(\phi)(\psi) - F(0)(\psi)) + F(0)(\psi)| \\ &\leq |F(\phi)(\psi) - F(0)(\psi)| + |F(0)(\psi)| \\ &\leq s \|\psi\|_{-\varrho,1} \|\phi - 0\|_{\varrho,0} + K \|\psi\|_{-\varrho,1} \\ &\leq (s \|\phi\|_{\varrho,0} + K) \|\psi\|_{-\varrho,1} \implies (17) \end{aligned}$$

Definition 5.1 (F_ϱ)**.** With (18) and (19) we can now extend F to a Lipschitz continuous function F_ϱ from $H_\varrho^0(\mathbb{R}) \otimes H$ to $H_\varrho^{-1}(\mathbb{R}) \otimes H$ with Lipschitz constant $s < 1$.

With this motivation we can formulate the first result that is well-known from the classical theory of differential equations:

Theorem 5.2 (Picard-Lindelőf, [Theorem 3.2 Kal+14, p. 13])**.** *Let $\varrho \in \mathbb{R}_{>0}$, $s \in (0, 1)$ and let $F\colon C_c^\infty(\mathbb{R}; H) \to C_{c+}^\infty(\mathbb{R}; H)'$ such that the estimates (18) and (19) hold for ϱ. Then there exists a uniquely determined $u \in H_\varrho^0(\mathbb{R}) \otimes H$ such that*

$$\partial_\varrho u = F_\varrho(u) \textit{ holds in } H_\varrho^{-1}(\mathbb{R}) \otimes H.$$

5.2. Higher regularity

A slightly different approach can be taken starting from (16)

$$\partial_\varrho u = F(u) \tag{20}$$

by looking for a solution for $\partial_\varrho u$ instead of u directly. Since ∂_ϱ is unitary this is an equivalent problem.

Call $v = \partial_\varrho u$. Then (16) becomes

$$v = F(\partial_\varrho^{-1} v).$$

In order to use the Theorem 5.2 of Picard-Lindelőf write this as

$$\partial_\varrho v = \partial_\varrho F(\partial_\varrho^{-1} v)$$

That means that $v \in H^0_\varrho(\mathbb{R}) \otimes H$, hence $F\colon H^1_\varrho(\mathbb{R}) \otimes H \to H^0_\varrho(\mathbb{R}) \otimes H$. Since ∂_ϱ is unitary, F must be a contraction. We stay with the assumption that F is given as a function from $C_c^\infty(\mathbb{R}; H)$ to $C_{c+}^\infty(\mathbb{R}; H)'$. In order to view functionals on $C_{c+}^\infty(\mathbb{R}; H)$ as elements of $H^0_\varrho(\mathbb{R}) \otimes H$ we identify $H^0_\varrho(\mathbb{R}) \otimes H$ with $(H^0_{-\varrho}(\mathbb{R}) \otimes H)^*$ as in 4.3: $\phi \in H^0_\varrho(\mathbb{R}) \otimes H$ is identified with $H^0_{-\varrho}(\mathbb{R}) \otimes H \ni \psi \mapsto \langle \psi, \phi \rangle_{0,0}$.

With this understanding we arrive at the conditions

$$\text{there is } K \in \mathbb{R}, \text{ such that } |F(0)(\psi)| \leq K \|\psi\|_{-\varrho,0} \text{ for all } \psi \in C_{c+}^\infty(\mathbb{R}; H) \tag{21}$$

to ensure that the values of F can be continuously extended to $H^0_\varrho(\mathbb{R}) \otimes H$. Secondly there exists $s \in (0,1)$ such that for all $u, w \in C_c^\infty(\mathbb{R}; H)$ and all $\psi \in C_{c+}^\infty(\mathbb{R}; H)$

$$|F(u)(\psi) - F(w)(\psi)| \leq s \|\psi\|_{-\varrho,0} \|u - w\|_{\varrho,1} \tag{22}$$

to ensure Lipschitz continuity.

Corollary 5.3 (Picard-Lindelőf with higher regularity, [Corollary 3.3 Kal+14, p. 13])**.** *Let $\varrho \in \mathbb{R}_{>0}$, $s \in (0,1)$ and let $F\colon C_c^\infty(\mathbb{R}; H) \to C_{c+}^\infty(\mathbb{R}; H)'$ be such that there is $K \in \mathbb{R}_{>0}$ such that for all $u, w \in C_c^\infty(\mathbb{R}; H)$ and $\psi \in C_{c+}^\infty(\mathbb{R}; H)$ we have*

$$|F(0)(\psi)| \leq K \|\psi\|_{-\varrho,0} \text{ and } |F(u)(\psi) - F(w)(\psi)| \leq s \|\psi\|_{-\varrho,0} \|u - w\|_{\varrho,1}.$$

Denote by $F_\varrho\colon H^1_\varrho(\mathbb{R}) \otimes H \to H^0_\varrho(\mathbb{R}) \otimes H$ the strictly contractive extension of F. Then there is a unique $u \in H^1_\varrho(\mathbb{R}) \otimes H$ satisfying

$$\partial_\varrho u = F_\varrho(u) \text{ in } H^0_\varrho(\mathbb{R}) \otimes H.$$

What have we gained with this corollary? The solution is not only existing and unique but also once (weakly) differentiable. In order to achieve this we need different conditions on the equation. F must map into $H^0_\varrho(\mathbb{R}) \otimes H$, not only $H^{-1}_\varrho(\mathbb{R}) \otimes H$ which is a stronger condition but on the other hand, in Corollary 5.3 F must be Lipschitz continuous with respect to the $H^1_\varrho(\mathbb{R}) \otimes H$ norm in contrast to the $H^0_\varrho(\mathbb{R}) \otimes H$ norm in Theorem 5.2. This is a weaker condition since

$$\|u\|_{\varrho,0} = \left\|\partial_\varrho^{-1} \partial_\varrho u\right\|_{\varrho,0} \leq \frac{1}{\varrho} \|u\|_{\varrho,1} \text{ for } u \in H^1_\varrho(\mathbb{R}) \otimes H \tag{23}$$

whereas an estimate in the other direction is not possible because ∂_ϱ is unbounded. There is no reason to assume that F is a contraction in differential equations of interest. With the equation (23) this problem can be solved though if we choose ϱ big enough. In the following theorem we cover the two cases at once but with slightly different arguments.

Corollary 5.4. *Let $k \in \{0,1\}$, $C \in \mathbb{R}_{>0}$, $\varrho > C$ and let $F\colon C_c^\infty(\mathbb{R}; H) \to C_{c+}^\infty(\mathbb{R}; H)'$ be such that there exists $K \in \mathbb{R}_{>0}$ such that for all $u, w \in C_c^\infty(\mathbb{R}; H)$ and $\psi \in C_{c+}^\infty(\mathbb{R}; H)$ we have*

$$|F(0)(\psi)| \leq K \|\psi\|_{-\varrho,-k} \text{ and } |F(u)(\psi) - F(w)(\psi)| \leq C \|\psi\|_{-\varrho,-k} \|u - w\|_{\varrho,k}.$$

We denote by $F_\varrho\colon H^k_\varrho(\mathbb{R})\otimes H \to H^k_\varrho(\mathbb{R})\otimes H$ the unique continuous extension of F. Then there is a unique $u \in H^{k+1}_\varrho(\mathbb{R})\otimes H$ with

$$\partial_\varrho u = F_\varrho(u) \text{ in } H^k_\varrho(\mathbb{R})\otimes H.$$

Proof. Case $k=0$. By (23) the continuity requirement for F implies for $u, w \in C_c^\infty(\mathbb{R}; H)$, $\psi \in C_{c+}^\infty(\mathbb{R}; H)$

$$|F(0)(\psi)| \leq K\frac{1}{\varrho}\,\|\psi\|_{-\varrho,1} \text{ and } |F(u)(\psi) - F(w)(\psi)| \leq \underbrace{C\frac{1}{\varrho}}_{=:s<1}\,\|\psi\|_{-\varrho,1}\,\|u-w\|_{\varrho,0}\,.$$

By Theorem 5.2 F has a contractive extension $\hat{F}_\varrho\colon H^0_\varrho(\mathbb{R})\otimes H \to H^{-1}_\varrho(\mathbb{R})\otimes H$ and a unique solution $u \in H^0_\varrho(\mathbb{R})\otimes H$ such that

$$\partial_\varrho u = \hat{F}_\varrho(u) \text{ in } H^{-1}_\varrho(\mathbb{R})\otimes H.$$

Since F_ϱ is the unique continuous extension of F on $H^0_\varrho(\mathbb{R})\otimes H$ this equation holds in $H^0_\varrho(\mathbb{R})\otimes H$ which implies that $u \in \mathrm{D}(\partial_\varrho) = H^1_\varrho(\mathbb{R})\otimes H$.

Case $k=1$. In order to consider an element of $C_{c+}^\infty(\mathbb{R}; H)'$ as an element of $H^1_\varrho(\mathbb{R})\otimes H$, consider the canonical identification of a Hilbert space with its bidual:

$$H^1_\varrho(\mathbb{R})\otimes H \cong (H^1_\varrho(\mathbb{R})\otimes H^*)^* = (H^{-1}_{-\varrho}(\mathbb{R})\otimes H)^*$$
$$\text{with the embedding } C_{c+}^\infty(\mathbb{R}; H)' \subset H^0_{-\varrho}(\mathbb{R})\otimes H \subset H^{-1}_{-\varrho}(\mathbb{R})\otimes H.$$

The assumption guarantees that $F(u)$ can be extended to an element of $H^1_\varrho(\mathbb{R})\otimes H$ for all $u \in C_c^\infty(\mathbb{R}; H)$. By (13) in Lemma 4.3 the continuity requirement for F implies for $u, w \in C_c^\infty(\mathbb{R}; H)$, $\psi \in C_{c+}^\infty(\mathbb{R}; H)$

$$|F(0)(\psi)| \leq K\frac{1}{\varrho}\,\|\psi\|_{-\varrho,0} \text{ and } |F(u)(\psi) - F(w)(\psi)| \leq \underbrace{C\frac{1}{\varrho}}_{=:s<1}\,\|\psi\|_{-\varrho,0}\,\|u-w\|_{\varrho,0}\,.$$

By Theorem 5.3 F has a contractive extension $\hat{F}_\varrho\colon H^1_\varrho(\mathbb{R})\otimes H \to H^0_\varrho(\mathbb{R})\otimes H$ and a unique solution $u \in H^1_\varrho(\mathbb{R})\otimes H$ such that

$$\partial_\varrho u = \hat{F}_\varrho(u) \text{ in } H^0_\varrho(\mathbb{R})\otimes H.$$

Since F_ϱ is the unique continuous extension of F on $H^1_\varrho(\mathbb{R})\otimes H$ this equation holds in $H^1_\varrho(\mathbb{R})\otimes H$ which implies that $\partial_\varrho u \in H^1_\varrho(\mathbb{R})\otimes H$, hence $u \in \mathrm{D}\left(\partial_\varrho^2\right)$ which could be called $H^2_\varrho(\mathbb{R})\otimes H$: u is twice weakly differentiable. □

6. Causality

In the previous sections we have seen conditions for ordinary differential equations to be uniquely solvable. Natural systems in physics have another property that should be modeled within this framework: causality. A solution of an equation up to any time should only depend on the previous behavior but not on the behavior in the future.

We will see that the conditions on the equation must again be tightened: where we needed an Lipschitz estimate for some weight ϱ we now will need that this estimate holds for eventually all weights, that is there is some ϱ_0 such that for all $\varrho > \varrho_0$ the estimate holds.

In the end of this section we will consider causality for the solution operator that maps the function F to the solution of the corresponding equation but on the way we will also show causality for $\partial_\varrho^{-1} F$, hence the definition regards any operator.

Definition 6.1 (Causality, [Definition 4.1 Kal+14, p. 18])**.** Let X, Y be Hilbert spaces, $\varrho \in \mathbb{R}$. A mapping

$$W\colon H_\varrho^0(\mathbb{R}) \otimes X \supseteq \mathrm{D}(W) \to H_\varrho^0(\mathbb{R}) \otimes Y$$

is called *causal* if for all $a \in \mathbb{R}$, $x, y \in \mathrm{D}(W)$

$$\chi_{\mathbb{R}_{<a}}(m)(x-y) = 0 \implies \chi_{\mathbb{R}_{<a}}(m)(W(x) - W(y)) = 0 :$$

if two arguments do not differ for arguments $\geq a$ then the images do not differ as well. This is equivalent to (for all $a \in \mathbb{R}$)

$$\chi_{\mathbb{R}_{<a}}(m)W = \chi_{\mathbb{R}_{<a}}(m)W\chi_{\mathbb{R}_{<a}}(m).$$

Here $\chi_{\mathbb{R}_{<a}}(m)$ is to be understood as in Definition 3.2:

$$(\chi_{\mathbb{R}_{<a}}(m)x)(t) = \chi_{\mathbb{R}_{<a}}(t)x(t) = \begin{cases} x(t) & t < a \\ 0 & t \geq a \end{cases}.$$

Equivalence. "$\implies$": For any $x \in \mathrm{D}(W)$ we obviously have $\chi_{\mathbb{R}_{<a}}(m)(x - \chi_{\mathbb{R}_{<a}}(m)x) = 0$ and hence with the linearity of $\chi_{\mathbb{R}_{<a}}(m)$

$$\chi_{\mathbb{R}_{<a}}(m)(W(x) - W(\chi_{\mathbb{R}_{<a}}(m)x)) = 0 \implies \chi_{\mathbb{R}_{<a}}(m)W(x) = W(\chi_{\mathbb{R}_{<a}}(m)x).$$

"$\impliedby$": Let $x, y \in \mathrm{D}(W)$. $\chi_{\mathbb{R}_{<a}}(m)(x-y) = 0$ implies $\chi_{\mathbb{R}_{<a}}(m)x = \chi_{\mathbb{R}_{<a}}(m)y$ by linearity of $\chi_{\mathbb{R}_{<a}}(m)$. Hence

$$\begin{aligned}\chi_{\mathbb{R}_{<a}}(m)(W(x) - W(y)) &= \chi_{\mathbb{R}_{<a}}(m)W(\chi_{\mathbb{R}_{<a}}(m)x) - \chi_{\mathbb{R}_{<a}}(m)W(\chi_{\mathbb{R}_{<a}}(m)y) \\ &= \chi_{\mathbb{R}_{<a}}(m)W(\chi_{\mathbb{R}_{<a}}(m)x) - \chi_{\mathbb{R}_{<a}}(m)W(\chi_{\mathbb{R}_{<a}}(m)x) = 0. \quad \square\end{aligned}$$

The first step to prove the causality of the solution operator is to prove the causality of the contraction $\partial_\varrho^{-1} F_\varrho$. For this proof we need to relate contraction on weighted spaces for different weights and hence need a ϱ-independent formulation of ∂_ϱ^{-1}. This is already commented on at the Definition 3.1 of the test space $C_{c+}^\infty(\mathbb{R}; H)$.

Definition 6.2 ($\int$ on $C^\infty_{c+}(\mathbb{R};H)$, [Definition 4.3 Kal+14, p. 18])**.** Let $w \in C^\infty_{c+}(\mathbb{R};H)'$. Then define

$$\int_{-\infty}^{\cdot} w \colon C^\infty_{c+}(\mathbb{R};H) \to \mathbb{C} \colon \psi \to w\left(\int_{\cdot}^{\infty} \psi\right).$$

As usual we identify w with an element of $H^{-1}_\varrho(\mathbb{R}) \otimes H$ if it continuous with respect to the $H^{-1}_\varrho(\mathbb{R}) \otimes H$-norm. Then, as we expected, we have for $\psi \in C^\infty_{c+}(\mathbb{R};H)$ and $\varrho > 0$:

$$\left(\int_{-\infty}^{\cdot} w\right)(\psi) = w\left(\int_{\cdot}^{\infty} \psi\right) \overset{(11)}{=} w(-\partial_{-\varrho}^{-1}\psi) \overset{4.5}{=} \partial_\varrho^{-1} w(\psi).$$

Theorem 6.3 (Causality of Lipschitz function, [Theorem 6.3 Kal+14, p. 19])**.** *Let $\varrho_0 \in \mathbb{R}_{>0}$, $F \colon C^\infty_c(\mathbb{R};H) \to C^\infty_{c+}(\mathbb{R};H)'$ be such that there exists $L \in \mathbb{R}_{\geq 0}$ and for each $\varrho > \varrho_0$ there exist $K \in \mathbb{R}_{>0}$ such that for all $u, w \in C^\infty_c(\mathbb{R};H)$ and $\psi \in C^\infty_{c+}(\mathbb{R};H)$*

$$|F(0)(\psi)| \leq K \|\psi\|_{-\varrho,1} \text{ and } |F(u)(\psi) - F(w)(\psi)| \leq L \|\psi\|_{-\varrho,1} \|u-w\|_{\varrho,0}.$$

Then for $\varrho \in \mathbb{R}_{>\varrho_0}$ the mapping

$$\partial_\varrho^{-1} F_\varrho \colon H^0_\varrho(\mathbb{R}) \otimes H \to H^0_\varrho(\mathbb{R}) \otimes H$$

is causal. Here F_ϱ is the continuous extension of F to a mapping $H^0_\varrho(\mathbb{R}) \otimes H \to H^{-1}_\varrho(\mathbb{R}) \otimes H$ as before.

Note that the assumption on F becomes different to the existence theorems by assuming the Lipschitz estimate for eventually all weights but with an arbitrary Lipschitz constant.

Proof. First we start with smooth functions for which we have formulas to calculate with. Let $\varrho \in \mathbb{R}_{>\varrho_0}$, $v \in C^\infty_c(\mathbb{R};H)$, $a \in \mathbb{R}$. We have to show

$$\chi_{\mathbb{R}_{<a}}(m)\partial_\varrho^{-1} F_\varrho v = \chi_{\mathbb{R}_{<a}}(m)\partial_\varrho^{-1} F_\varrho \chi_{\mathbb{R}_{<a}}(m) v.$$

$\partial_\varrho^{-1} F_\varrho$ maps to $H^0_\varrho(\mathbb{R}) \otimes H$ but when written out we only have the formulation as a mapping to $H^0_{-\varrho}(\mathbb{R}) \otimes H^*$ from Theorem 4.5. What does it mean to apply $\chi_{\mathbb{R}_{<a}}(m)$ to an element of $H^0_{-\varrho}(\mathbb{R}) \otimes H^*$? For $w \in H^0_\varrho(\mathbb{R}) \otimes H \cong H^0_{-\varrho}(\mathbb{R}) \otimes H^*$, $\psi \in H^0_{-\varrho}(\mathbb{R}) \otimes H$ we have

$$\begin{aligned}(\chi_{\mathbb{R}_{<a}}(m)w)\psi &= \left\langle \chi_{\mathbb{R}_{<a}}(m)w, \psi\right\rangle_{0,0} = \int_{\mathbb{R}} \left\langle \chi_{\mathbb{R}_{<a}}(m)w(x), \psi(x)\right\rangle_H dx \\ &= \int_{\mathbb{R}_{<a}} \langle w(x), \psi(x)\rangle_H \, dx = \int_{\mathbb{R}} \left\langle w(x), \chi_{\mathbb{R}_{<a}}(m)\psi(x)\right\rangle_H dx \\ &= \left\langle w, \chi_{\mathbb{R}_{<a}}(m)\psi\right\rangle_{0,0} = w(\chi_{\mathbb{R}_{<a}}(m)\psi)\end{aligned}$$

Hence for checking $\chi_{\mathbb{R}_{<a}}(m)w = \chi_{\mathbb{R}_{<a}}(m)\tilde{w}$ we only have to test the equality on

$$\psi \in C^\infty_c(\mathbb{R};H) \subset H^0_{-\varrho}(\mathbb{R}) \otimes H \text{ with } \sup\operatorname{supp}\psi \leq a.$$

$$\text{We have to show: } \partial_\varrho^{-1} F_\varrho(v)(\psi) = \partial_\varrho^{-1} F_\varrho(\chi_{\mathbb{R}_{<a}}(m)v)(\psi). \tag{24}$$

$\chi_{\mathbb{R}_{<a}}(m)v$ is not smooth in a, hence cannot plugged into F. In order to approximate $\chi_{\mathbb{R}_{<a}}(m)v$ with smooth functions, we replace $\chi_{\mathbb{R}_{<a}}(m)$ with $\phi(m)$ with $\phi \in C^\infty(\mathbb{R}; H)$ and $\sup \operatorname{supp} \phi \le a$. Then for any $\eta \in \mathbb{R}_{\ge \varrho_0}$

$$\begin{aligned}
\left|\partial_\varrho^{-1} F_\varrho(v)(\psi) - \partial_\varrho^{-1} F_\varrho(\phi(m)v)(\psi)\right| &\overset{\text{Def. 6.2}}{=} \left| \left(\int_{-\infty}^{\cdot} F(v) \right)(\psi) - \left(\int_{-\infty}^{\cdot} F(\phi(m)v) \right)(\psi) \right| \\
&= \left|\partial_\eta^{-1} F_\eta(v)(\psi) - \partial_\eta^{-1} F_\eta(\phi(m)v)(\psi)\right| \\
&\qquad \text{since the continuity estimate holds for any} \\
&\qquad \eta \ge \varrho_0 \text{ and hence } F_\eta \text{ exists} \\
&= \left| F_\eta(v)(-\partial_{-\eta}^{-1}\psi) - F_\eta(\phi(m)v)(-\partial_{-\eta}^{-1}\psi)\right| \\
&\qquad (\text{Def. 4.5: } \partial_\eta^{-1} \text{ on } H_\eta^{-1}(\mathbb{R}) \otimes H) \\
&\le L \left\| -\partial_{-\eta}^{-1}\psi \right\|_{-\eta,1} \left\| v - \phi(m)v \right\|_{\eta,0} \\
&\qquad \text{since } F_\eta \text{ is Lipschitz continuous} \\
&= L \left\| \psi \right\|_{-\eta,0} \left\| v - \phi(m)v \right\|_{\eta,0} \\
&\le L \left\| \psi \right\|_{0,0} \exp(\eta a) \left\| v - \phi(m)v \right\|_{\eta,0}
\end{aligned}$$

since

$$\begin{aligned}
\|\psi\|_{-\eta,0}^2 &= \int_{\mathbb{R}} \|\psi(t)\|_H^2 \exp(2\eta t)\, dt = \int_{\infty}^{a} \|\psi(t)\|_H^2 \exp(2\eta t)\, dt \\
&\le \int_{\mathbb{R}} \|\psi(t)\|_H^2 \, dt \exp(2\eta a) = \|\psi\|_{0,0}^2 \exp(\eta a)^2 < \infty \text{ since } \psi \in C_c^\infty(\mathbb{R}; H).
\end{aligned}$$

By continuity we can again replace ϕ by $\chi_{\mathbb{R}_{<a}}$. To see that consider any ϕ that agrees with $\chi_{\mathbb{R}_{<a}}$ except for some small interval $(a-\varepsilon, a)$. There v is bounded. Consider the difference

$$\|(\phi - \chi_{\mathbb{R}<a})(m)v\|_{\eta,0} \le \left\|\phi - \chi_{\mathbb{R}_{<a}}\right\|_{\eta,0} \underbrace{\sup_{x \in (a-\varepsilon)} \|v(x)\|_H}_{<\infty}.$$

By choosing ϕ sufficiently close to $\chi_{\mathbb{R}_{<a}}$, we can approximate $\chi_{\mathbb{R}_{<a}}(m)v$ arbitrarily close with $\phi(m)v$.

We get

$$\begin{aligned}
\left|\partial_\varrho^{-1} F_\varrho(v)(\psi) - \partial_\varrho^{-1} F_\varrho(\phi(m)v)(\psi)\right| &\le L \|\psi\|_{0,0} \exp(\eta a) \left\| v - \chi_{\mathbb{R}_{<a}}(m)v \right\|_{\eta,0} \\
&= L \|\psi\|_{0,0} \exp(\eta a) \left\| \chi_{\mathbb{R}_{\ge a}}(m)v \right\|_{\eta,0} \\
&= L \|\psi\|_{0,0} \left(\int_a^\infty \|v(t)\|_H^2 \exp(-2\eta t) \exp(2\eta a)\, dt \right)^{\frac{1}{2}}
\end{aligned}$$

$$= L \|\psi\|_{0,0} \left(\int_0^\infty \|v(s+a)\|_H^2 \underbrace{\exp(-2\eta s)}_{\to 0,\, \eta\to\infty} \, ds \right)^{\frac{1}{2}}$$

$$(s = t - a)$$

$$\to 0 \text{ for } \eta \to \infty \text{ by the dominated convergence theorem}$$

As we discussed for (24) in the beginning, this means

$$\chi_{\mathbb{R}_{<a}}(m) \left(\partial_\varrho^{-1} F_\varrho(v) - \partial_\varrho^{-1} F_\varrho(\chi_{\mathbb{R}_{<a}}(m)v) \right) = 0.$$

Since $\chi_{\mathbb{R}_{<a}}(m)$ and $\partial_\varrho^{-1} F_\varrho$ are continuous, this also holds for arbitrary $v \in H_\varrho^0(\mathbb{R}) \otimes H$. □

6.1. Solution independent of the weight

All theorems about existence, uniqueness and causality depended on some choice of the search space characterized by the weight ϱ. Since the original differential equation is independent of ϱ it is important to ask if the solution depends on ϱ. First we check that the contraction $\partial_\varrho^{-1} F_\varrho$ does not depend on ϱ:

Lemma 6.4. *Let $G \colon C_c^\infty(\mathbb{R}; H) \to C_{c+}^\infty(\mathbb{R}; H)'$ such that G has continuous extensions to functions $G_k \colon H_{\varrho_k}^0(\mathbb{R}) \otimes H \to H_{\varrho_k}^{-1}(\mathbb{R}) \otimes H$ for $\varrho_2 \geq \varrho_1$. For $v_k \in H_{\varrho_k}^0(\mathbb{R}) \otimes H$ ($k \in \{1, 2\}$) and $a \in \mathbb{R}$ we have*

$$\chi_{\mathbb{R}_{>a}}(m) v_1 \in H_{\varrho_2}^0(\mathbb{R}) \otimes H$$
$$\chi_{\mathbb{R}_{<a}}(m) v_2 \in H_{\varrho_1}^0(\mathbb{R}) \otimes H.$$

For $w \in H_{\varrho_1}^0(\mathbb{R}) \otimes H \cap H_{\varrho_2}^0(\mathbb{R}) \otimes H$

$$G_1 w = G_2 w \in C_{c+}^\infty(\mathbb{R}; H)'.$$

Proof. To show the first statements consider the definitions for any $a \in \mathbb{R}$:

$$\begin{aligned} \left\|\chi_{\mathbb{R}_{>a}}(m) v_1\right\|_{\varrho_2,0}^2 &= \int_{\mathbb{R}} \left\|\chi_{\mathbb{R}_{>a}}(m) v_1(t)\right\|_H^2 \exp(-2\varrho_2 t) \, dt \\ &= \int_a^\infty \|v_1(t)\|_H^2 \exp(-2\varrho_1 t) \exp(-2(\underbrace{\varrho_2 - \varrho_1}_{\geq 0})t) \, dt \\ &\leq \int_a^\infty \|v_1(t)\|_H^2 \exp(-2\varrho_2 t) \, dt \, \exp(-2(\varrho_2 - \varrho_1)a) \\ &= \|v_1\|_{\varrho_1,0}^2 \underbrace{\exp(-2a(\varrho_2 - \varrho_1))}_{:= C_{a,2} < \infty} \end{aligned} \tag{25}$$

and in the same way

$$\left\|\chi_{\mathbb{R}_{<a}}(m) v_2\right\|_{\varrho_1,0}^2 \leq \|v_2\|_{\varrho_2,0}^2 \underbrace{\exp(-2a(\varrho_1 - \varrho_2))}_{:= C_{a,1} < \infty}. \tag{26}$$

Those two estimates together show that the norms of $H^0_{\varrho_1}(\mathbb{R}) \otimes H$ and $H^0_{\varrho_2}(\mathbb{R}) \otimes H$ are comparable if we restrict ourselves to functions with compact support. Hence it is useful to define $w_a := \chi_{[-a,a]} w$ $(a > 0)$ for $w \in H^0_{\varrho_1}(\mathbb{R}) \otimes H \cap H^0_{\varrho_2}(\mathbb{R}) \otimes H$.

Since G_1 and G_2 are defined as the closure of G we have to find sequences of test functions converging to w in both norms. As a first step consider any sequence $(\varphi_{a,k})_{k\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}; H)$ that converges to $\chi_{[-a,a]} w$ in $H^0_{\varrho_1}(\mathbb{R}) \otimes H$. Because of the comparability of the norms it also converges to $\chi_{[-a,a]} w$ in $H^0_{\varrho_2}(\mathbb{R}) \otimes H$. (Choose $(\varphi_{a,k})_{k\in\mathbb{N}}$ such that $\operatorname{supp} \varphi_{a,k} \subseteq [-a-1, a+1]$.)

Since $\exp(-2\varrho_i m) \|w_a\|_H^2 \nearrow \exp(-2\varrho_i m) \|w\|_H^2$ for $i \in \{1, 2\}$ and $a \to \infty$ the monotone convergence theorem implies convergence of $w_a \to w$ for $a \to \infty$ in both norms. Hence the diagonal sequence $(\varphi_{k,k})_{k\in\mathbb{N}} \to w$ in both norms for $k \to \infty$.

The assumed boundedness for G carry over to G_1 and G_2. We can use it to show that the equality $G_1(\varphi_{k,k}) = G(\varphi_{k,k}) = G_2(\varphi_{k,k})$ implies $G_1 w = G_2 w$: In order to consider $G_2(w)$ and $G_1(w)$ as elements of a common space, consider elements of $H^{-1}_{\varrho_i}(\mathbb{R}) \otimes H$ $(i \in \{1, 2\})$ as elements of $C_{c+}^\infty(\mathbb{R}; H)'$ by restriction. (Note that the restriction is injective since $C_{c+}^\infty(\mathbb{R}; H)$ is dense in $H^1_{\varrho_i}(\mathbb{R}) \otimes H$ $(i \in \{1, 2\})$.) Let $\psi \in C_{c+}^\infty(\mathbb{R}; H)$. Then

$$\begin{aligned}
|(G_2(w) - G_1(w))(\psi)| &\le \left|(G_2(w) - G_2(\varphi_{k,k}))(\psi)\right| + \left|(G_2(\varphi_{k,k}) - G_1(\varphi_{k,k}))(\psi)\right| \\
&\quad + \left|(G_1(\varphi_{k,k}) - G_1(w))(\psi)\right| \\
&\le \underbrace{\left\|G_2(w) - G_2(\varphi_{k,k})\right\|_{\varrho_2,-1}}_{\to 0,\ k\to\infty} \|\psi\|_{-\varrho_2,1} + 0 \\
&\quad + \underbrace{\left\|G_1(\varphi_{k,k}) - G_1(w)\right\|_{\varrho_1,-1}}_{\to 0,\ k\to\infty} \|\psi\|_{-\varrho_1,1} \to 0 \text{ as } k \to \infty
\end{aligned}$$

because G_1 and G_2 are continuous and $\varphi_{k,k} \to w$ in both norms. □

From here it is a small step to show that the solution is independent of the weight.

Theorem 6.5 (Solution independent of the weight, [Theorem 4.6 Kal+14, p. 19])**.** *Let $\varrho_0 \in \mathbb{R}_{>0}$, $F\colon C_c^\infty(\mathbb{R}; H) \to C_{c+}^\infty(\mathbb{R}; H)'$ be such that for each $\varrho > \varrho_0$ there exist $K_\varrho \in \mathbb{R}_{>0}$ and $s_\varrho \in (0, 1)$ such that for all $u, w \in C_c^\infty(\mathbb{R}; H)$ and $\psi \in C_{c+}^\infty(\mathbb{R}; H)$*

$$|F(0)(\psi)| \le K_\varrho \|\psi\|_{-\varrho,1} \text{ and } |F(u)(\psi) - F(w)(\psi)| \le s_\varrho \|\psi\|_{-\varrho,1} \|u - w\|_{\varrho,0}.$$

For $\varrho_2 \ge \varrho_1 \ge \varrho_0$ let $w_{\varrho_k} \in H^0_{\varrho_k}(\mathbb{R}) \otimes H$ $(k \in \{1, 2\})$ be the solution to

$$\partial_{\varrho_k} u = F_{\varrho_k}(u) \in H^{-1}_{\varrho_k}(\mathbb{R}) \otimes H \quad (k \in \{1, 2\}).$$

Then the solutions are the same:

$$w_1 = w_2 \in H^0_{\varrho_1}(\mathbb{R}) \otimes H \cap H^0_{\varrho_2}(\mathbb{R}) \otimes H$$

Proof. Let $G_k := \partial_{\varrho_k}^{-1} F_{\varrho_k}$ $(k \in \{0, 1, 2\})$ for simplicity.

The solution w_1 is unique in $H^0_{\varrho_1}(\mathbb{R}) \otimes H$ for a single ϱ because G_1 is a contraction. In analogy to the proof for the uniqueness of fixed points of the contraction G_1 we would like to use the (wrong) inequality

$$\|w_1 - w_2\|_{\varrho_1,0} = \|G_1(w_1) - G_1(w_2)\|_{\varrho_1,0} \leq s_{\varrho_1} \|w_1 - w_2\|_{\varrho_1,0} \text{ with } s_{\varrho_1} < 1.$$

We cannot use it directly since w_2 is not necessarily in the domain of G_1 and if it was we do not know if $G_1(w_2) = w_2$. In the previous Lemma 6.4 we have seen though that $\chi_{\mathbb{R}_{<a}}(m)w_2 \in H^0_{\varrho_1}(\mathbb{R}) \otimes H$ for any $a \in \mathbb{R}$. So we can consider

$$G_1(\chi_{\mathbb{R}_{<a}}(m)w_1) - G_1(\chi_{\mathbb{R}_{<a}}(m)w_2) = G_1(\chi_{\mathbb{R}_{<a}}(m)w_1) - G_2(\chi_{\mathbb{R}_{<a}}(m)w_2) \quad \text{by Lemma 6.4.}$$

In general $G_1(\chi_{\mathbb{R}_{<a}}(m)w_1)$ could be anything but due to causality and Theorem 6.3 we know

$$\chi_{\mathbb{R}_{<a}}(m)G_k(\chi_{\mathbb{R}_{<a}}(m)w_k) = \chi_{\mathbb{R}_{<a}}(m)G_k(w_k) \quad (k = 1, 2).$$

Hence

$$\begin{aligned}
\chi_{\mathbb{R}_{<a}}(m) &\left(G_1(\chi_{\mathbb{R}_{<a}}(m)w_1) - G_1(\chi_{\mathbb{R}_{<a}}(m)w_2)\right) \\
&= \chi_{\mathbb{R}_{<a}}(m) \left(G_1(\chi_{\mathbb{R}_{<a}}(m)w_1) - G_2(\chi_{\mathbb{R}_{<a}}(m)w_2)\right) \\
&= \chi_{\mathbb{R}_{<a}}(m)G_1(w_1) - \chi_{\mathbb{R}_{<a}}(m)G_2(w_2) \\
&= \chi_{\mathbb{R}_{<a}}(m)(w_1 - w_2)
\end{aligned}$$

since w_1 and w_2 are fixed points of G_1 and G_2 respectively. Now we can use the contraction argument from the beginning:

$$\begin{aligned}
\left\|\chi_{\mathbb{R}_{<a}}(m)(w_1 - w_2)\right\|_{\varrho_1,0} &= \left\|\chi_{\mathbb{R}_{<a}}(m) \left(G_1(\chi_{\mathbb{R}_{<a}}(m)w_1) - G_1(\chi_{\mathbb{R}_{<a}}(m)w_2)\right)\right\|_{\varrho_1,0} \\
&\leq \left\|G_1(\chi_{\mathbb{R}_{<a}}(m)w_1) - G_1(\chi_{\mathbb{R}_{<a}}(m)w_2)\right\|_{\varrho_1,0} \\
&\leq s_{\varrho_1} \left\|\chi_{\mathbb{R}_{<a}}(m)(w_1 - w_2)\right\| \quad \text{with } s_{\varrho_1} < 1.
\end{aligned}$$

Only for $\chi_{\mathbb{R}_{<a}}(m)w_1 = \chi_{\mathbb{R}_{<a}}(m)w_2$ this is no contradiction. So w_1 and w_2 agree on $(-\infty, a)$ for $a \in \mathbb{R}$ but a was arbitrary, so $w_1 = w_2$. □

6.2. Causality of the solution operator

The goal of this section is to prove the causality of the solution operator. That is, the solution up to any time t does not change when changing the "data" (i. e. the right hand side) up to the time t. But in order to talk about the solution operator one needs to give the possible right hand sides ("the F's") a name:

Definition 6.6 (Eventually contracting, [Definition 4.7 Kal+14, p. 20])**.** Let $\mathrm{Con}_{\mathrm{ev}}$ be the set of all mappings $F\colon C^\infty_c(\mathbb{R}; H) \to C^\infty_{c+}(\mathbb{R}; H)'$ such that there exists an ϱ such that for all $\eta \in \mathbb{R}_{\geq\varrho}$ there is $K \in \mathbb{R}_{>0}$ and $s \in (0, 1)$ such that for all $u, w \in C^\infty_c(\mathbb{R}; H)$ and $\psi \in C^\infty_{c+}(\mathbb{R}; H)$ we have

$$|F(0)(\psi)| \leq K \|\psi\|_{-\eta,1} \text{ and } |F(u)(\psi) - F(w)(\psi)| \leq s \|\psi\|_{-\eta,1} \|u - w\|_{\eta,0}$$

Remark (Eventually). The functions in $\mathrm{Con}_{\mathrm{ev}}$ are called "eventually contracting" because they are contraction from some weight onward but for any finite interval they might not be.

For $F \in \mathrm{Con}_{\mathrm{ev}}$ we have the results of Theorem 5.2 for existence and uniqueness of a solution and know by Theorem 6.5 that the solution does not depend on the choice of the weighted space we solve the equation in. This enables us to write the relation of the right hand side and the solution as an well-defined operator:

Definition 6.7 (Solution operator)**.** Define

$$S\colon \mathrm{Con}_{\mathrm{ev}} \to \bigcup_{\varrho>0} H^0_\varrho(\mathbb{R}) \otimes H$$
$$F \mapsto S(F) \text{ such that } S(F) = \partial_\varrho^{-1} F(S(F)) \text{ for sufficiantly large } \varrho.$$

As emphasized before another desired property of the solution operator is causality. So far causality is only defined for operators of the type

$$W\colon H^0_\varrho(\mathbb{R}) \otimes X \supseteq \mathrm{D}(W) \to H^0_\varrho(\mathbb{R}) \otimes Y$$

(compare Definition 6.1). In order to to say that F and G in $\mathrm{Con}_{\mathrm{ev}}$ do not differ for small arguments (i. e. $< a$ for some $a \in \mathbb{R}$) we can apply (i. e. concatenate) $\chi_{\mathbb{R}_{<a}}(m)$ with F and G. But since F and G are mapping into distribution spaces (spaces of functionals, not functions) we also include the ∂_ϱ^{-1} that is causal and brings us back into a space of functions.

Theorem 6.8 (Causality of solving, [Theorem 4.8 Kal+14, p. 21])**.** *Let $\varrho \in \mathbb{R}_{>0}$, $a \in \mathbb{R}$. Let $F, G \in \mathrm{Con}_{ev}$ such that*

$$\chi_{\mathbb{R}_{<a}}(m)\partial_\varrho^{-1} F = \chi_{\mathbb{R}_{<a}}(m)\partial_\varrho^{-1} G. \tag{27}$$

Then $\chi_{\mathbb{R}_{<a}}(m)S(F) = \chi_{\mathbb{R}_{<a}}(m)S(G)$.

Proof. Let $x = S(F)$ and $y = S(G)$. We need to show $\chi_{\mathbb{R}_{<a}}(m)x = \chi_{\mathbb{R}_{<a}}(m)y$. The solutions to the regarded differential equation are x and y but what does this imply for $\chi_{\mathbb{R}_{<a}}(m)x$ and $\chi_{\mathbb{R}_{<a}}(m)y$? Here the causality of $\partial_\varrho^{-1}F$ and $\partial_\varrho^{-1}G$ comes into play in order to relate the truncated solutions to the equations:

$$\begin{aligned} x &= \partial_\varrho^{-1} F(x) \\ \implies \chi_{\mathbb{R}_{<a}}(m)x &= \chi_{\mathbb{R}_{<a}}(m)\partial_\varrho^{-1} F(x) \\ \overset{6.3}{\implies} \chi_{\mathbb{R}_{<a}}(m)x &= \chi_{\mathbb{R}_{<a}}(m)\partial_\varrho^{-1} F(\chi_{\mathbb{R}_{<a}}(m)x) \\ &\overset{(27)}{=} \chi_{\mathbb{R}_{<a}}(m)\partial_\varrho^{-1} G(\chi_{\mathbb{R}_{<a}}(m)x) \end{aligned}$$

and similarly $\chi_{\mathbb{R}_{<a}}(m)y = \chi_{\mathbb{R}_{<a}}(m)\partial_\varrho^{-1} G(\chi_{\mathbb{R}_{<a}}(m)y)$

In other words, $\chi_{\mathbb{R}_{<a}}(m)x$ and $\chi_{\mathbb{R}_{<a}}(m)y$ are both fixed points of the mapping

$$\chi_{\mathbb{R}_{<a}}(m)\partial_\varrho^{-1} G$$

which is a strictly contracting since $\partial_\varrho^{-1}G$ is strictly contracting and $\chi_{\mathbb{R}_{<a}}(m)$ is contracting. Hence $\chi_{\mathbb{R}_{<a}}(m)x = \chi_{\mathbb{R}_{<a}}(m)y$. ☐

6.3. Delay

Causality is able to capture the notion of "the result does not depend on the future" but it does not distinguish between actually depending on the past or not. Intuitively one would not consider the differential x' in a typical ordinary differential equation of the type

$$x'(t) = f(t, x(t))$$

to depend neither on the past nor on the future of x. Hence we need the analogue concept to causality for "the result does not depend on the past".

Definition 6.9 (Amnesic, [Definition 4.9 Kal+14, p. 22])**.** Let X, Y be Hilbert spaces, $\varrho \in \mathbb{R}$. A mapping

$$W\colon H^0_\varrho(\mathbb{R}) \supseteq \mathrm{D}(W) \otimes X \to H^0_\varrho(\mathbb{R}) \otimes Y$$

is called *causal* if for all $a \in \mathbb{R}$, $x, y \in \mathrm{D}(W)$

$$\chi_{\mathbb{R}_{>a}}(m)(x - y) = 0 \implies \chi_{\mathbb{R}_{>a}}(m)(W(x) - W(y)) = 0 :$$

if two arguments do not differ for arguments greater than a then the images do not differ as well. This is equivalent to (for all $a \in \mathbb{R}$)

$$\chi_{\mathbb{R}_{>a}}(m)W = \chi_{\mathbb{R}_{>a}}(m)W\chi_{\mathbb{R}_{>a}}(m).$$

If a mapping W is not amnesic, we say that W *has memory* or *has delay*.

In many ways causal and amnesic operators have the same or *dual* properties: there is nothing special about $\mathbb{R}_{>a}$ in comparison to $\mathbb{R}_{<a}$ but ∂_ϱ^{-1} is causal for $\varrho > 0$ but amnesic for $\varrho < 0$ as one can see in (10) and (11). Hence the same theory as developed above for amnesic operators would replace $\varrho > 0$ by $\varrho < 0$.

When we showed the causality of F we actually considered $\partial_\varrho^{-1} F$ because F maps into a space of functionals ($C^\infty_{c+}(\mathbb{R}; H)'$ or $H^{-1}_\varrho(\mathbb{R})$) and hence we needed ∂_ϱ^{-1} to get into a function space. This is justified since ∂_ϱ^{-1} is causal.

For defining amnesic or respectively having delay for differential equations we have the same issue and we cannot use ∂_ϱ^{-1} since it is not amnesic but we can use a duality similar to the results in Section 4.1. For the following calculation note that taking inverses and adjoints commute and due to unitarity $\exp(-\varrho m)^{-1} = \exp(-\varrho m)^*$

$$\begin{aligned}
\left(\partial_\varrho^{-1}\right)^* &= \left(\left(\exp(-\varrho m)^{-1}(\partial + \varrho)\exp(-\varrho m)\right)^{-1}\right)^* \\
&= \left(\exp(-\varrho m)^{-1}(\partial + \varrho)^{-1}\exp(-\varrho m)\right)^* \\
&= \exp(-\varrho m)^*((\partial + \varrho)^*)^{-1}\exp(-\varrho m) \\
&= \exp(-\varrho m)^{-1}(-\partial + \varrho)^{-1}\exp(-\varrho m) \\
&= -\exp(-\varrho m)^{-1}\exp(\varrho m)\partial_{-\varrho}^{-1}\exp(\varrho m)^{-1}\exp(-\varrho m).
\end{aligned}$$

By the Formula (11) in Corollary 3.6 $\partial_{-\varrho}^{-1}$ is amnesic and hence $\left(\partial_\varrho^{-1}\right)^*$ is an amnesic operator from $H^{-1}_\varrho(\mathbb{R}) \otimes H \to H^0_\varrho(\mathbb{R}) \otimes H$. This allows us to *define* what a differential equation with delay *is*.

Definition 6.10 (Delay differential equation, [Definition 4.11 Kal+14, pp. 22 sq.])**.** Let $\varrho > 0$ and $F \in \mathrm{Con}_{\mathrm{ev}}$. A differential equation of the form (16), i. e.,

$$\partial_\varrho u = F(u)$$

is called a *delay differential equation* if $\left(\partial_\varrho^{-1}\right)^* F$ has delay.

7. Applications

7.1. Nemitzki operator

In the introduction of Section 6.3 we already mentioned the intuitively expected result that a “typical” ordinary differential is causal and amnesic. The formalisation is called *Nemitzki operator*:

Definition 7.1 (Nemitzki operator, [Example 4.10 Kal+14, p. 22])**.** Let $f\colon \mathbb{R}\times H \to H$, $\varrho_0 \in \mathbb{R}_{>0}$ such that f is measurable and uniformly, with respect to the first (time) variable, Lipschitz continuous, i. e., there exists $L > 0$ such that for all $t \in \mathbb{R}$ and $x, y \in H$ we have

$$\left\|f(t,x) - f(t,y)\right\|_H \leq L \left\|x - y\right\|_H$$

and additionally $f(\cdot, 0) \in \bigcap_{\varrho\geq\varrho_0} H^0_\varrho(\mathbb{R}) \otimes H$. Those properties ensure, that

$$F_\varrho\colon H^0_\varrho(\mathbb{R}) \otimes H \to H^0_\varrho(\mathbb{R}) \otimes H\colon u \mapsto (t \mapsto f(t, u(t)))$$

is well-defined for all $\varrho \geq \varrho_0$ and is the right hand side of a uniquely solvable differential equation for all $\varrho > \max\{L, \varrho_0\}$. F_ϱ is called a *Nemitzki operator*.

Remark (Generalisation)*.* Note that the term *Nemitzki operator* can be defined in a more general setting but in our case we need those stricter conditions to apply the developed theory.

Corollary 7.2 (Nemitzki is amnesic)**.** *Let f, L, ϱ and F_ϱ be as in the Definition 7.1. Then F_ϱ is amnesic and causal.*

Proof. Let $a \in \mathbb{R}$, $u, v \in H^0_\varrho(\mathbb{R}) \otimes H$ such that

$$\chi_{\mathbb{R}_{>a}}(m)u = \chi_{\mathbb{R}_{>a}}(m)v.$$

Then for all $t \in \mathbb{R}$

$$\begin{aligned}
\chi_{\mathbb{R}_{>a}}(m)(F_\varrho(u) - F_\varrho(v)) &= \chi_{\mathbb{R}_{>a}}(m)((t \mapsto f(t,u(t))) - (t \mapsto f(t,v(t)))) \\
\implies \chi_{\mathbb{R}_{>a}}(m)(F_\varrho(u) - F_\varrho(v))(t) & \\
&= \begin{cases} \underbrace{\chi_{\mathbb{R}_{>a}}(m)(t)}_{=0}(f(t,u(t)) - f(t,v(t))) & (t \leq a) \\ \underbrace{\chi_{\mathbb{R}_{>a}}(m)(t)}_{=1}\,(f(t,\underbrace{(\chi_{\mathbb{R}_{>a}}(m)u)(t)}_{=u(t)}) & \\ \quad -f(t,\underbrace{(\chi_{\mathbb{R}_{>a}}(m)v)(t)}_{=v(t)})) & (t > a) \end{cases} \\
&= 0
\end{aligned}$$

where $\mathbb{R}_{>a}$ can also be replaced by $\mathbb{R}_{<a}$. (Of course the two cases for t must be adjusted accordingly.) Hence F_ϱ is causal and amnesic. ☐

7.2. Time translation

For formulating delay differential equations we will consider translation of functions as an operator.

Definition 7.3 (Time translation, [Example 2.12 Kal+14, p. 12])**.** Let $\varrho \in \mathbb{R}_{>0}$, $h \in \mathbb{R}$ and $u \in H_\varrho^0(\mathbb{R}) \otimes H$. Define

$$\tau_h u := u(\cdot + h).$$

The operator $\tau_h \in L(H_\varrho^0(\mathbb{R}) \otimes H, H_\varrho^0(\mathbb{R}) \otimes H)$ is called *time translation operator*. For $h < 0$, τ_h is called the *delay operator*.

Lemma 7.4. *τ_h is continuous with $\|\tau_h\|_{L(H_\varrho^0(\mathbb{R})\otimes H, H_\varrho^0(\mathbb{R})\otimes H)} = \exp(h\varrho)$.*

Proof. Let $u \in H_\varrho^0(\mathbb{R}) \otimes H$ with $\|u\|_{H_\varrho^0(\mathbb{R})\otimes H} = 1$. Then

$$\begin{aligned}\|\tau_h u\|_{\varrho,0}^2 &= \int_{\mathbb{R}} \|u(x+h)\|_H^2 \exp(-2\varrho x)\, dx = \int_{\mathbb{R}} \|u(x)\|_H^2 \exp(-2\varrho(x-h))\, dx \\ &= \int_{\mathbb{R}} \|u(x)\|_H^2 \exp(-2\varrho x) \exp(-2\varrho(-h))\, dx = \exp(\varrho h)^2 \|u\|_{H_\varrho^0(\mathbb{R})\otimes H}^2\end{aligned}$$

Hence $\|\tau_h u\|_{\varrho,0} = \exp(\varrho h) \|u\|_{\varrho,0}$. □

7.3. Discrete Delay

Finally we have reached the point of considering the typical delay differential equations and see how they fit into the framework developed so far. A *delay differential equation with discrete delay* can typically be written as

$$x'(t) = f(x(t+h_1), x(t+h_2), x(t+h_3), ..., x(t+h_N))$$

with pairwise distinct delays $0 \geq h_1 > h_2 > ... > h_N$ $(N \in \mathbb{N})$. Here it comes in handy to split the right hand side into two factors. The first one is

$$\begin{aligned}&\Theta \colon C_c^\infty(\mathbb{R}; H) \to C_c^\infty(\mathbb{R}; H^N) \subseteq \bigcap_{\eta\in\mathbb{R}_{>0}} H_\eta^0(\mathbb{R}) \otimes H^N \\ &\Theta x = (\tau_{h_1} x, \tau_{h_2} x, ..., \tau_{h_N} x)\end{aligned}$$

that encapsulates the pasts of the argument. (Note that in Definition 7.3 we only defined τ_h for $H_\varrho^0(\mathbb{R}) \otimes H$ spaces but it is obvious how to similarly define time translation on arbitrary functions defined on the whole real line.)

The second factor is

$$\Phi \colon C_c^\infty(\mathbb{R}; H^N) \to C_{c+}^\infty(\mathbb{R}; H)'.$$

With this notation we can calculate the Lipschitz constant of Θ: As shown in Lemma 7.4 the time translation is continuous with

$$\|\tau_h\|_{L(H_\varrho^0(\mathbb{R})\otimes H, H_\varrho^0(\mathbb{R})\otimes H)} = \exp(h\varrho).$$

To use that for Θ note that $H_\varrho^0(\mathbb{R}) \otimes H^N \cong (H_\varrho^0(\mathbb{R}) \otimes H)^N$. Then

$$\begin{aligned}
\|\Theta\|^2_{L(H_\varrho^0(\mathbb{R})\otimes H, H_\varrho^0(\mathbb{R})\otimes H^N)} &\leq \sum_{k=1}^{N} \left\|\tau_{h_k}\right\|^2_{L(H_\varrho^0(\mathbb{R})\otimes H, H_\varrho^0(\mathbb{R})\otimes H)} \\
&= \exp(2\varrho \underbrace{h_1}_{\leq 0}) + \sum_{k=2}^{N} \exp(2\varrho \underbrace{h_k}_{\leq h_2 < h_1 \leq 0}) \\
&\leq \exp(2\varrho h_1) + (N-1)\exp(2\varrho \underbrace{h_2}_{<0}) \\
&\begin{cases} \searrow 0 + 0 & h_1 < 0 \\ \searrow 1 + 0 & h_1 = 0 \end{cases} \quad (\varrho \to \infty)
\end{aligned}$$

Since $\|\Phi \circ \Theta\|_{\text{Lip}} \leq \|\Phi\|_{\text{Lip}} \|\Theta\|_{\text{Lip}}$ we also get two different conditions on Φ depending on the decision if the right hand side solely depends on the past or not:

Theorem 7.5 (Discrete delay, [Theorem 5.8 Kal+14, p. 31])**.** *Let $N \in \mathbb{N}$, let $0 \geq h_1 > h_2 > ... > h_N \in \mathbb{R}_{\leq 0}$ be the discrete delays, $s \in (0,1)$ if $h_1 = 0$ or $s \in \mathbb{R}_{>0}$ otherwise. Let $\varrho_0 \in \mathbb{R}_{>0}$ and $\Phi\colon C_c^\infty(\mathbb{R}, H^N) \to C_{c+}^\infty(\mathbb{R}; H)'$ such that for all $\varrho \in \mathbb{R}_{>\varrho_0}$ there is $K \in \mathbb{R}_{>0}$ such that for all $u, w \in C_c^\infty(\mathbb{R}, H^N)$ and $\psi \in C_{c+}^\infty(\mathbb{R}; H)$ we have*

$$|\Phi(0)(\psi)| \leq K \|\psi\|_{-\varrho,1} \ \text{ and } \ |\Phi(u)(\psi) - \Phi(w)(\psi)| \leq s \|\psi\|_{-\varrho,1} \|u - w\|_{\varrho,0}.$$

Call Φ_ϱ and Θ_ϱ the appropriate continuous extensions. Then for large enough ϱ the equation

$$\partial_\varrho u = \Phi_\varrho(\Theta_\varrho u)$$

has a unique solution $u \in H_\varrho^0(\mathbb{R})$ and the solution operator is causal.

Proof. As considered above the theorem we can choose ϱ large enough for

$$\|\Phi \circ \Theta\|_{\text{Lip}} \leq \|\Phi\|_{\text{Lip}} \|\Theta\|_{\text{Lip}} < s \cdot \frac{1}{s} = 1 \text{ since } \frac{1}{s} > 1 \text{ for } h_1 = 0 \text{ and } \frac{1}{s} > 0 \text{ for } h_1 < 0$$

By Theorem 5.2 of Picard-Lindelőf this implies the unique solution of the considered equation with causality of the solution operator guaranteed by Theorem 6.8. □

7.4. Continuous delay

A similar approach as for discrete delay can be taken to consider the second type of typical delay differential equations which are equations with continuous delay. The right hand side should not only depend on some points in the past but on the entire history. The most typical way of formulating this is (compare [Example 5.11 Kal+14, p. 33])

$$u'(t) = \int_{-\infty}^{0} h(t, \theta, \tau_\theta^- u(t))\, d\theta$$

with a suitable $h\colon \mathbb{R} \times \mathbb{R}_{<0} \times H \to H$. What "suitable" means we will see after applying the solution theory.

Here τ^- maps a function to its past:

$$\begin{aligned} \tau^- \colon H^{\mathbb{R}} &\to \left(H^{\mathbb{R}_{<0}}\right)^{\mathbb{R}} \\ \tau_\theta^- u(t) &= u(\theta + t) \quad \theta \in \mathbb{R}_{<0}, t \in \mathbb{R} \end{aligned}$$

In order to treat this case in the developed manner we generalise that to the factorisation $F = \Phi \circ \Theta$ with

$$\begin{aligned} \Theta\colon C_c^\infty(\mathbb{R}) &\to \bigcap_{\eta \in \mathbb{R}_{>0}} H_\eta^0(\mathbb{R}) \otimes L_2(\mathbb{R}_{<0}; H) \\ \Theta\colon u &\mapsto \tau^- u \\ \text{and } \Phi\colon \bigcap_{\eta \in \mathbb{R}_{>0}} H_\eta^0(\mathbb{R}) &\otimes L_2(\mathbb{R}_{<0}; H) \to C_{c+}^\infty(\mathbb{R}; H)' \end{aligned}$$

In the example above we get for Φ

$$\Phi(U) = \left(t \mapsto \underbrace{\int_{-\infty}^{0} h(t, \theta, U_\theta(t))\, d\theta}_{=: g(t, U(t))} \right)$$

where $g\colon \mathbb{R} \times L_2(\mathbb{R}_{<0}; H) \to H$ is an intermediate step of abstraction: a function that takes a time and a past until this time.

As in the previous Section 7.3 we need the Lipschitz constant of Θ in order to find the necessary Lipschitz condition for Φ. Since Θ is linear, $\|\Theta\|_{\mathrm{Lip}}$ is the operator norm of Hilbert space operators. Let $u \in C_c^\infty(\mathbb{R}; H)$. Then we compute:

$$\begin{aligned} \|\Theta(u)\|_{\varrho,0}^2 &= \int_{\mathbb{R}} \|\tau^- u(t)\|_{L_2(\mathbb{R}_{<0};H)}^2 \exp(-2\varrho t)\, dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}_{<0}} \|u(t+\theta)\|_H^2 \, d\theta \, \exp(-2\varrho t)\, dt \\ &= \int_{\mathbb{R}_{<0}} \int_{\mathbb{R}} \|u(t+\theta)\|_H^2 \exp(-2\varrho t)\, dt \, d\theta && \text{Fubini} \\ &= \int_{\mathbb{R}_{<0}} \int_{\mathbb{R}} \|u(s)\|_H^2 \exp(-2\varrho s)\, ds \, \exp(2\varrho\theta)\, d\theta && s = t + \theta \\ &= \int_{\mathbb{R}_{<0}} \|u\|_{\varrho,0}^2 \exp(2\varrho\theta)\, d\theta \end{aligned}$$

$$
\begin{aligned}
&= \|u\|_{\varrho,0}^{2} \int_{\mathbb{R}_{<0}} \exp(2\varrho\theta)\, d\theta \\
&= \|u\|_{\varrho,0}^{2} \left(\frac{1}{2\varrho} \exp(2\varrho\theta)\Big|_{-\infty}^{0} \right) = \frac{1}{2\varrho} \|u\|_{\varrho,0}^{2}
\end{aligned}
$$

$$
\implies \|\Theta\|_{\mathrm{Lip}} = \frac{1}{\sqrt{2\varrho}}
$$

The application of the theorem of Fubini is permitted since the double integral is finite as shown and all integrands are non-negative.

Hence with growing ϱ the Lipschitz constant of Φ can grow as well. So we get to the following conclusion.

Theorem 7.6 (Continuous delay, [Theorem 5.10 Kal+14, p. 32])**.** *Let*

$$
\Phi\colon \bigcap_{\eta\in\mathbb{R}_{>0}} H_\eta^0(\mathbb{R}) \otimes L_2(\mathbb{R}_{<0}; H) \to C_{c+}^\infty(\mathbb{R}; H)'
$$

such that there exists ϱ_0 such that for all $\varrho \in \mathbb{R}_{>\varrho_0}$, there exists $K \in \mathbb{R}_{>0}$ and $s < \sqrt{2\varrho}$ such that for all $u, w \in \bigcap_{\eta\in\mathbb{R}_{>0}} H_\eta^0(\mathbb{R}) \otimes L_2(\mathbb{R}_{<0}H;)$ and $\psi \in C_{c+}^\infty(\mathbb{R}; H)$ we have

$$
|\Phi(0)(\psi)| \leq K \|\psi\|_{-\varrho,1} \quad \text{and} \quad |\Phi(u)(\psi) - \Phi(w)(\psi)| \leq s \|\psi\|_{-\varrho,1} \|u - w\|_{\varrho,0}.
$$

Let Φ_ϱ be the Lipschitz continuous extension of Φ to a mapping from $H_\varrho^0(\mathbb{R}) \otimes L_2(\mathbb{R}_{<0}; H)$ to $H_\varrho^{-1}(\mathbb{R}) \otimes H$. Then the equation

$$
\partial_\varrho u = \Phi_\varrho(\tau^- u)
$$

has a unique solution and the solution operator is causal.

Remark (Possible Lipschitz constant)*.* One possibility to describe s depending on ϱ is $C\varrho^r$ with $r \in (0, \frac{1}{2})$ and $C \in \mathbb{R}_{>0}$. In order to have $C\varrho^r < \sqrt{2\varrho}$ calculate:

$$
C\varrho^r < (2\varrho)^{\frac{1}{2}} \iff \frac{C}{\sqrt{2}} < \varrho^{-r+\frac{1}{2}} \iff \varrho > \left(\frac{C}{\sqrt{2}}\right)^{\frac{1}{\frac{1}{2}-r}} = \left(\frac{C}{\sqrt{2}}\right)^{\frac{2}{1-2r}}.
$$

So choose $\varrho_0 > \left(\frac{C}{\sqrt{2}}\right)^{\frac{2}{r-2}}$.

Example 7.7 ([Example 5.11 (a) Kal+14, p. 33])*.* Going back to the beginning of the section we should ask what the condition for Φ means for g.

The first one:

$$
\begin{aligned}
& \left|\Phi(0)(\psi)\right| \leq K \left\|\psi\right\|_{-\varrho,1} \\
\text{implies for } g\text{:} \quad & \left|\int_{\mathbb{R}} \left\langle g(t,0), \psi(t)\right\rangle_H \, dt\right| = \left|g(\cdot,0)(\psi)\right| \leq K \left\|\psi\right\|_{-\varrho,1} \\
\text{where} \quad & \left|\int_{\mathbb{R}} \left\langle g(t,0), \psi(t)\right\rangle_H \, dt\right| = \left|\int_{\mathbb{R}} \left\langle g(t,0), \partial_{-\varrho}^{-1}\partial_{-\varrho}\psi(t)\right\rangle_H \, dt\right| \\
& \stackrel{14}{=} \left|\int_{\mathbb{R}} \left\langle \partial_{\varrho}^{-1} g(t,0), \partial_{-\varrho}\psi(t)\right\rangle_H \, dt\right| \\
& \leq \left\|\partial_{\varrho}^{-1} g(\cdot,0)\right\|_{\varrho,0} \left\|\partial_{-\varrho}\psi(t)\right\|_{-\varrho,0} \\
& \stackrel{3.5}{\leq} \frac{1}{\varrho} \left\|g(\cdot,0)\right\|_{\varrho,0} \left\|\psi\right\|_{-\varrho,1}
\end{aligned}
$$

Hence we need to impose on g, that there is a $\varrho_0 \in \mathbb{R}_{>0}$ such that for all $\varrho \in \mathbb{R}_{>\varrho_0}$ $\left\|g(\cdot,0)\right\|_{\varrho,0}$ is finite.

Similarly we can formulate the Lipschitz condition without the test function and in $H_\varrho^0(\mathbb{R}) \otimes H$. Let $u, w \in \bigcap_{\eta\in\mathbb{R}_{>0}} H_\eta^0(\mathbb{R}) \otimes L_2(\mathbb{R}_{<0}; H)$.

$$
\left\|t \mapsto g(t,u(t)) - g(t,w(t))\right\|_{\varrho,0} \leq \varrho s \left\|u - w\right\|_{\varrho,0}
$$

where the extra ϱ comes from the step from $H_\varrho^{-1}(\mathbb{R}) \otimes H$ to $H_\varrho^0(\mathbb{R}) \otimes H$. Written with integrals, this is

$$
\int_{\mathbb{R}} \left\|g(t,u(t)) - g(t,w(t))\right\|_H^2 \exp(-2\varrho t)\, dt \;\leq \varrho^2 s^2 \int_{\mathbb{R}} \left\|u(t) - w(t)\right\|_H^2 \exp(-2\varrho t)\, dt
$$

One option for g to fulfill this estimate is a Lipschitz constant $L \in \mathbb{R}_{>0}$ such that we have for all $x, y \in L_2(\mathbb{R}_{>0}; H)$ and all $t \in \mathbb{R}$

$$
\left\|g(t,x) - g(t,y)\right\|_H \leq L \left\|x - y\right\|_{L_2(\mathbb{R}_{>0};H)}
$$

which would imply

$$
\begin{aligned}
\int_{\mathbb{R}} \left\|g(t,u(t)) - g(t,w(t))\right\|_H^2 \exp(-2\varrho t)\, dt \;&\leq \int_{\mathbb{R}} L \left\|u(t) - w(t)\right\|_H^2 \exp(-2\varrho t)\, dt \\
&= L \left\|u - w\right\|_{\varrho,0}^2 .
\end{aligned}
$$

This L must therefore fulfill

$$
L \leq \varrho^2 s^2 < \varrho^2 \cdot 2\varrho
$$

for sufficiently large ϱ. Since $2\varrho^3 \to \infty$, L can be chosen arbitrary.

In conclusion we have the following result.

Corollary 7.8. *Let* $g\colon \mathbb{R}\times L_2(\mathbb{R}_{<0};H)\to H$ *such that there exists* $L\in\mathbb{R}_{>0}$ *such that for all sufficiently large* ϱ *and all* $t\in\mathbb{R}$ *and* $x,y\in L_2(\mathbb{R}_{<0};H)$

$$\|g(\cdot,0)\|_{\varrho,0} \text{ is finite and } \|g(t,x)-g(t,y)\|_H \le L\,\|x-y\|_{L_2(\mathbb{R}_{<0};H)}.$$

Then the equation

$$\partial_\varrho u(t) = g(t,\tau^- u(t)) \quad (t\in\mathbb{R})$$

has a unique solution and the solution operator is causal.

If we now plug in h into the definition of g and look for suitable conditions on h the result is very bulky. So we go back to the beginning at the condition for F and assume h to be Lipschitz continuous with respect to the third argument.

Corollary 7.9 ([Example 5.11 (b) Kal+14, p. 33])**.** *Let* $\varrho_0\in\mathbb{R}_{>0}$,

$$\begin{aligned} h\colon \mathbb{R}\times\mathbb{R}_{<0}\times H &\to H \\ \text{with } h(t,\theta,0) &= 0 \\ \text{and } \|h(t,\theta,x)-h(t,\theta,y)\|_H &\le L(\theta)\,\|x-y\|_H \text{ for all } t\in\mathbb{R},\theta\in\mathbb{R}_{<0},x,y\in H \\ \text{with } L\colon \mathbb{R}_{<0} &\to \mathbb{R}_{>0} \\ \text{such that} \int_{-\infty}^0 L(\theta)\exp(\varrho\theta)\,d\theta &< \varrho \text{ for all } \varrho\ge\varrho_0. \end{aligned} \tag{28}$$

For example L *can be any constant function. Let*

$$\begin{aligned} F\colon C_c^\infty(\mathbb{R};H) &\to \bigcap_{\eta\in\mathbb{R}_{>\varrho_0}} H_\eta^0(\mathbb{R})\otimes H \subset C_c^\infty(\mathbb{R};H)' \\ F(u) &= \left(t\mapsto \int_{-\infty}^0 h(t,\theta,\tau_\theta^- u(t))\,d\theta\right). \end{aligned}$$

For all $\varrho\ge\varrho_0$ *let* F_ϱ *be the Lipschitz continuous extension of* F *to a mapping from* $H_\varrho^0(\mathbb{R})\otimes H$ *to* $H_\varrho^{-1}(\mathbb{R})\otimes H$*. Then the equation*

$$\partial_\varrho u = F_\varrho u \tag{29}$$

has a unique solution and the solution operator is causal.

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Proof. Let $\varrho\in\mathbb{R}_{\ge\varrho_0}$, $u,v\in C_c^\infty(\mathbb{R};H)$, $\psi\in C_{c+}^\infty(\mathbb{R};H)$. F_ϱ maps into $H_\varrho^{-1}(\mathbb{R})\otimes H$ since

$$|F(0)(\psi)| = \left|\int_{\mathbb{R}}\left\langle \int_{-\infty}^0 h(t,\theta,0)\,d\theta\,,\psi(t)\right\rangle_H dt\right| = \left|\langle 0,\psi\rangle_{0,0}\right| = 0.$$

Check the Lipschitz condition for F.

$$
\begin{aligned}
&|F(u)(\psi) - F(v)(\psi)| \\
&= \left| \int_{\mathbb{R}} \left\langle \int_{-\infty}^{0} h(t,\theta,\tau_\theta^- u(t))\, d\theta\,, \psi(t) \right\rangle_H - \left\langle \int_{-\infty}^{0} h(t,\theta,\tau_\theta^- v(t))\, d\theta\,, \psi(t) \right\rangle_H dt \right| \\
&= \left| \int_{\mathbb{R}} \int_{-\infty}^{0} \langle h(t,\theta,u(t+\theta)) - h(t,\theta,v(t+\theta)), \psi(t) \rangle_H \, d\theta \; dt \right|
\end{aligned}
$$

With Fubini this is

$$
\begin{aligned}
&= \left| \int_{-\infty}^{0} \int_{\mathbb{R}} \langle h(t,\theta,u(t+\theta)) - h(t,\theta,v(t+\theta)), \psi(t) \rangle_H \, dt \; d\theta \right| \\
&= \left| \int_{-\infty}^{0} \langle h(\cdot,\theta,u(\cdot+\theta)) - h(\cdot,\theta,v(\cdot+\theta)), \psi \rangle_{0,0} \; d\theta \right|
\end{aligned}
$$

By (14) and Cauchy-Schwarz inequality this can be estimated by

$$
\leq \int_{-\infty}^{0} \| t \mapsto h(t,\theta,u(t+\theta)) - h(t,\theta,v(t+\theta)) \|_{\varrho,-1} \|\psi\|_{-\varrho,1} \; d\theta
$$

By Corollary 3.5 estimate this with

$$
\begin{aligned}
&\leq \frac{1}{\varrho} \|\psi\|_{-\varrho,1} \int_{-\infty}^{0} \| t \mapsto h(t,\theta,u(t+\theta)) - h(t,\theta,v(t+\theta)) \|_{\varrho,0} \; d\theta \\
&= \frac{1}{\varrho} \|\psi\|_{-\varrho,1} \int_{-\infty}^{0} \left(\int_{\mathbb{R}} \| h(t,\theta,u(t+\theta)) - h(t,\theta,v(t+\theta)) \|_H^2 \exp(-2\varrho t)\, dt \right)^{\frac{1}{2}} d\theta \\
&\leq \frac{1}{\varrho} \|\psi\|_{-\varrho,1} \int_{-\infty}^{0} \left(\int_{\mathbb{R}} L(\theta)^2 \| u(t+\theta) - v(t+\theta) \|_H^2 \exp(-2\varrho t)\, dt \right)^{\frac{1}{2}} d\theta
\end{aligned}
$$

With the substitution $s = t + \theta$ this is

$$
\begin{aligned}
&= \frac{1}{\varrho} \|\psi\|_{-\varrho,1} \int_{-\infty}^{0} \left(\int_{\mathbb{R}} L(\theta)^2 \| u(s) - v(s) \|_H^2 \exp(-2\varrho(s-\theta))\, ds \right)^{\frac{1}{2}} d\theta \\
&= \frac{1}{\varrho} \|\psi\|_{-\varrho,1} \int_{-\infty}^{0} L(\theta) \|u - v\|_{\varrho,0} \exp(\varrho\theta)\, d\theta \\
&< \frac{1}{\varrho} \|\psi\|_{-\varrho,1} \, \varrho \, \|u - v\|_{\varrho,0} = \|\psi\|_{-\varrho,1} \|u - v\|_{\varrho,0}
\end{aligned}
$$

By Theorem 5.2 of Picard-Lindelőf (29) has a unique solution for all $\varrho > \varrho_0$ and Theorem 6.5 implies that the solution does not depend on ϱ. By Theorem 6.8 the solution operator is causal. □

A. Tensor products of Hilbert spaces

Ordinary differential equations are sometimes categorized into scalar equations and systems of equation. The difference is that the function searched for maps to $\mathbb{R}$ or rather $\mathbb{C}$ in our case for scalar equations and maps to $\mathbb{C}^n$ ($n \in \mathbb{N}$) for systems of equations. Explicitly calculating solutions for systems is of course more complicated but regarding the solution theory it hardly changes anything especially in the solution theory developed here. That is still true when we generalise further to any Hilbert space instead of the finite dimensional $\mathbb{C}^n$. One might even notice that while reading one easily overlooks that we are talking throughout about Hilbert space valued functions because everything works exactly like in $\mathbb{C}$. In order to get to this point we have to properly introduce $L_2(\mathbb{R}; H)$ for an Hilbert space H though.

In order to go from the accustomed space $L_2(\mathbb{R};\mathbb{C})$ to $L_2(\mathbb{R}; H)$ an equivalent construction with tensor products is used. Tensor products are not necessarily a common topic in linear algebra courses which might be supported by the difficulty to get a visual understanding of them. Therefore several different approaches were taken to define and understand tensor products. Here we use conjugate bilinear forms as it is done in [Tro11, A.1].

It is easily possible to skip most of this section, especially the proofs without any negative impact on the understanding of the rest of the thesis since everything works out just as expected. Important facts are:

- The definitions of pure tensors A.1 and the tensor product A.3 as the completion of the algebraic tensor product.
- Operators can be lifted as expected (Definition A.4).
- $L_2(\mathbb{R};\mathbb{C}) \otimes H \cong L_2(\mathbb{R}; H)$ (Theorem A.9)

A.1.

Definition A.1 (Algebraic Tensor (product)). Let H_1, H_2 be complex Hilbert spaces, $x \in H_1$, $y \in H_2$. Then we consider the conjugate bilinear continuous functional called *pure tensor*

$$\begin{aligned} x \otimes y \colon H_1 \times H_2 &\to \mathbb{C} \\ (\phi, \psi) &\mapsto \langle x, \phi \rangle_{H_1} \langle y, \psi \rangle_{H_2} . \end{aligned}$$

Remark (conjugate bilinear). Here the tensors are defined to be conjugate-bilinear (or antibilinear). That is: $(x \otimes y)(\alpha\phi, \beta\psi) = \alpha^*\beta^*(x \otimes y)(\phi, \psi)$ for any $\alpha, \beta \in \mathbb{C}$. An analogous definition could be done with bilinear functionals by putting x and y on the right side of the inner products. That might feel more familiar but it would also imply that the scalar multiplication in the tensor product would be a conjugate-linear, that is $\alpha(x \otimes y) = (\alpha^* x \otimes y)$. So one has to decide which linear structure feels familiar. Here the linear structure in the tensor product is chosen because we are mostly interested in the tensor product and the actual definition via functionals is not so much of interest.

Pure tensors are elements of the vector space of all conjugate bilinear functionals and therefore inherit the following linear structure.

$$\left(\sum_{i=1}^{n} a_i(x_i \otimes y_i)\right)(\phi, \psi) = \sum_{i=1}^{n} a_i(x_i \otimes y_i)(\phi, \psi) \quad (\phi \in H_1, \psi \in H_2)$$

for $x_i \in H_1$, $y_i \in H_2$, $a_i \in \mathbb{C}$ for $i \in \{1, ..., n\}$.

The linear combinations are also conjugate bilinear functionals. We call the space of these linear combinations

$$V_1 \overset{a}{\otimes} V_2 := \mathrm{Lin}\,\{x \otimes y \,|\, x \in V_1, y \in V_2\}$$

the *algebraic tensor product* of the subsets V_1 of H_1 and V_2 of H_2.

Remark. Note that every linear combination of pure tensors can be written as a sum of pure tensors. For $n \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$, $x_i \in H_1$, $y_i \in H_2$ $(i \in \{1, ..., n\})$:

$$\sum_{i=1}^{n} \alpha_i(x_i \otimes y_i) = \sum_{i=1}^{n} \underbrace{(\alpha_i x_i)}_{\in H_1} \otimes y_i \in H_1 \overset{a}{\otimes} H_2 \tag{30}$$

The inner products of H_1 and H_2 induce a sesquilinear mapping on the algebraic tensor product $H = H_1 \overset{a}{\otimes} H_2$. We define it on pure tensors and extend it sesquilinearly:

$$\begin{aligned} \langle x \otimes y, u \otimes v\rangle_{\otimes} &= (x \otimes y)(u, v) = \langle x, u\rangle_{H_1} \langle y, v\rangle_{H_2} = \langle u, x\rangle^*_{H_1} \langle v, y\rangle^*_{H_2} \\ &= (u \otimes v)(x, y)^* = \langle u \otimes v, x \otimes y\rangle^* \quad (x, u \in H_1, y, v \in H_2) \end{aligned} \tag{31}$$

The name "product" is justified because we get a copy of H_2 for every element of H_1: by fixing one particular x and $\phi \in H_1$ one gets the trivial embedding that maps any $y \in H_2$ to an conjugate-linear functional on H_2. Note that this space of conjugate-linear continuous functionals on an Hilbert space is isomorphic to H_2 itself.

Furthermore one sees that in the case of finite dimensional H_1 and H_2, we have $\dim(H_1 \otimes H_2) = \dim(H_1) \cdot \dim(H_2)$.

The sesquilinear form $\langle\cdot,\cdot\rangle_{\otimes}$ is denoted as if it was an inner product.

Lemma A.2. *For two Hilbert spaces* $\langle\cdot,\cdot\rangle_{\otimes}$ *defines an inner product on* $H_1 \overset{a}{\otimes} H_2$.

Proof. By definition $\langle\cdot,\cdot\rangle_{\otimes}$ is sesquilinear. As seen in (31) $\langle\cdot,\cdot\rangle_{\otimes}$ is conjugate symmetric on pure tensors. By sesquilinearity this extends to $H_1 \overset{a}{\otimes} H_2$.

The representation of elements of $H_1 \overset{a}{\otimes} H_2$ as linear combinations of pure tensors is not unique. Hence it must be checked that $\langle\cdot,\cdot\rangle_{\otimes}$ is well-defined. This is true since $H_1 \overset{a}{\otimes} H_2$ is a (linear) vector space and $\langle\cdot,\cdot\rangle_{\otimes}$ is defined to be sesquilinear. In detail, let

$$\sum_{i}^{n} a_i(x_i \otimes y_i) = \sum_{i}^{m} b_j(u_j \otimes v_j) \in H_1 \otimes H_2$$

with $n, m \in \mathbb{N}, \phi, x_i, u_j \in H_1, a_i, b_j \in \mathbb{C}, \psi, y_i, v_j \in H_2$ for $j \in \{1, ..., m\} i \in \{1, ..., n\}$.

Then for all $\phi \in H_1$ and $\psi \in H_2$

$$\left\langle \sum_i^n a_i(x_i \otimes y_i), \phi \otimes \psi \right\rangle_\otimes - \left\langle \sum_i^m b_j(u_j \otimes v_j), \phi \otimes \psi \right\rangle_\otimes$$
$$= \left(\sum_i^n a_i(x_i \otimes y_i)\right)(\phi, \psi) - \left(\sum_i^m b_j(u_j \otimes v_j)\right)(\phi, \psi)$$
$$= \left(\left(\sum_i^n a_i(x_i \otimes y_i)\right) - \left(\sum_i^m b_j(u_j \otimes v_j)\right)\right)(\phi, \psi)$$

By the linearity in the second argument, we have for all $w \in H_1 \overset{a}{\otimes} H_2$

$$\left\langle \sum_i^n a_i(x_i \otimes y_i), w \right\rangle_\otimes - \left\langle \sum_i^m b_j(u_j \otimes v_j), w \right\rangle_\otimes = 0.$$

This shows that $\langle \cdot, \cdot \rangle_\otimes$ is a function of the first argument for every second argument and by conjugate symmetry $\langle \cdot, \cdot \rangle_\otimes$ $\langle \cdot, \cdot \rangle_\otimes$ is a function of the second argument for every first argument.

For the positivity consider for $n \in \mathbb{N}$, $x_i \in H_1$, $y_i \in H_2$, $a_i \in \mathbb{C}$, $z_i = a_i x_i$ for $i \in \{1, ..., n\}$

$$\left\langle \sum_{i=1}^n a_i(x_i \otimes y_i), \sum_{j=1}^n a_j(x_j \otimes y_j) \right\rangle = \left\langle \sum_{i=1}^n (z_i \otimes y_i), \sum_{j=1}^n (z_j \otimes y_j) \right\rangle_\otimes$$
$$= \sum_{i=1}^n \sum_{j=1}^n \langle z_i, z_j \rangle_{H_1} \langle y_i, y_j \rangle_{H_2}$$

Note that the (so called *Gramian*) matrices $G_1 = (\langle z_i, z_j \rangle)_{i,j})$ and $G_2 = (\langle y_i, y_j \rangle)_{i,j}$ are hermitian since $\langle \cdot, \cdot \rangle_{H_1}$ and $\langle \cdot, \cdot \rangle_{H_2}$ is conjugate symmetric. Hence they can be diagonalised. G_1 and G_2 are positive semi-definite since $\langle \cdot, \cdot \rangle_{H_1}$ and $\langle \cdot, \cdot \rangle_{H_2}$ are positive definite. Positive semi-definite diagonal matrices have a square root and hence G_1 and G_2 have *hermitian* square roots. Call them $A_s = \left(a^s_{ij}\right)_{i,j}$ $(s \in \{1, 2\})$. With them the sum can be rewritten as (all sums going from 1 to n)

$$\sum_i \sum_j \sum_{k_1} a^1_{ik_1} a^1_{k_1 j} \left(\sum_{k_2} a^2_{ik_2} k_2 j^2\right)$$
$$= \sum_{k_1} \sum_{k_2} \left(\sum_i a^1_{ik_1} a^2_{ik_2}\right) \left(\sum_j a^1_{k_1 j} a^2_{k_2 j}\right) \qquad \text{(Rearranging sums)}$$
$$= \sum_{k_1} \sum_{k_2} \left(\sum_i a^1_{ik_1} a^2_{ik_2}\right) \left(\sum_j a^1_{jk_1} a^2_{jk_2}\right)^* \qquad (A_1,\ A_2 \text{ hermitian})$$

$$
\begin{aligned}
&= \sum_{k_1} \sum_{k_2} \left| \sum_i a^1_{ik_1} a^2_{ik_2} \right|^2 && (cc^* = |c|^2,\ c \in \mathbb{C}) \\
&\geq 0.
\end{aligned}
$$

To show, that $\langle \cdot, \cdot \rangle_\otimes$ is positive definite (not only semi-definite as we just showed), consider any $a \in H_1 \overset{a}{\otimes} H_2$ with $\langle a, a \rangle_\otimes = 0$ and a pure tensor $\phi \otimes \psi \in H_1 \overset{a}{\otimes} H_2$. Then by the Cauchy-Schwarz-Inequality that holds for all symmetric sesquilinear forms:

$$
|a(\phi, \psi)| = \left| \langle a, \phi \otimes \psi \rangle_\otimes \right| \leq \sqrt{\langle a, a \rangle_\otimes} \sqrt{\langle \phi \otimes \psi, \phi \otimes \psi \rangle_\otimes} = 0.
$$

Thus, $a = 0$ (as a function). □

So far everything was pure linear algebra. We are interested in (infinite dimensional) Hilbert spaces. Hence the question arises if the (algebraic) tensor product of two Hilbert spaces is again complete, that is it is an Hilbert space. Since we only allowed finite linear combinations it might not surprise that $H_1 \overset{a}{\otimes} H_2$ is in general not an Hilbert space. That is the reason why we consider the completion.

Definition A.3 (Tensor product)**.** Let H_1 and H_2 be Hilbert spaces. Let $H_1 \otimes H_2$ be the completion of $H_1 \overset{a}{\otimes} H_2$ with respect to the inner product $\langle \cdot, \cdot \rangle_\otimes$. The inner product of this Hilbert space is denoted by $\langle \cdot, \cdot \rangle_{H_1 \otimes H_2}$.

A.1. Operators on tensor products

We study operators on Hilbert spaces. Hence we have to know how operators can be lifted to tensor products (also see [Kal+14, Remark 2.9 (d)]). For pure tensors and hence for linear combinations it is obvious how to define it. Following is the proof that this is indeed well-defined and can be extended to the completion.

Definition A.4 (Algebraic Tensorproduct of linear operators)**.** Let H_1, H_2, K_1, K_2 be complex Hilbert spaces and $A\colon H_1 \supseteq \mathrm{D}(A) \to K_1$, $B\colon H_2 \supset \mathrm{D}(B) \to K_2$ be linear operators. Then define the *algebraic tensor product*

$$
\begin{aligned}
A \overset{a}{\otimes} B\colon H_1 \otimes H_2 \supset \mathrm{D}(A) \overset{a}{\otimes} \mathrm{D}(B) &\to K_1 \otimes K_2 \\
x \otimes y &\mapsto Ax \otimes By
\end{aligned}
$$

and extend it linearly. Then $A \overset{a}{\otimes} B$ is a well-defined linear mapping.

Proof. For clarification note that here $A \overset{a}{\otimes} B$ is considered as a subspace of $(H_1 \otimes H_2) \times (K_1 \otimes K_2)$ (i. e. the graph of $A \overset{a}{\otimes} B$).

$A \overset{a}{\otimes} B$ is linear. Therefore it is enough to show that $(0, w) \in A \overset{a}{\otimes} B$ implies $w = 0$. Let

$$
w = \sum_{i=1}^n (A\phi_i \otimes B\psi_i) \text{ with } \sum_{i=1}^n (\phi_i \otimes \psi_i) = 0
$$

where $\phi_i \in \mathrm{D}(A)$, $\psi_i \in \mathrm{D}(B)$ $(i = 1, ..., n)$.

The sets $\{\phi_i \,|\, i \in \{1, ..., n\}\}$ and $\{\psi_i \,|\, i \in \{1, ..., n\}\}$ are not necessarily linear independent. Hence choose a linear independent set $\left\{x_j \in \mathrm{D}(A) \,\middle|\, j \in \{1, ..., k\}\right\}$ such that all ϕ_i, and a linear independent set $\{y_l \in \mathrm{D}(B) \,|\, l \in \{1, ..., m\}\}$ such that all ψ_i can be represented as:

$$\phi_i = \sum_{j=1}^{k} \beta_j^i x_j, \quad \psi_i = \sum_{l=1}^{m} \gamma_l^i y_l \qquad i \in \{1, ..., n\}.$$

Then

$$\begin{aligned} w = \sum_{i=1}^{n} (A\phi_i \otimes B\psi_i) &= \sum_{i=1}^{n} A \left(\sum_{j=1}^{k} \beta_j^i x_j \right) \otimes B \left(\sum_{l=1}^{m} \gamma_l^i y_l \right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{k} \sum_{l=1}^{m} \beta_j^i \gamma_l^i \left(A x_j \otimes B y_l \right) \end{aligned} \tag{32}$$

and

$$0 = \sum_{i=1}^{n} (\phi_i \otimes \psi_i) = \sum_{i=1}^{n} \sum_{j=1}^{k} \sum_{l=1}^{m} \beta_j^i \gamma_l^i \left(x_j \otimes y_l \right). \tag{33}$$

For proving $w = 0$ we need to show that products of the form $\beta_j^i \gamma_l^i$ in (32) all vanish. By (33) it suffices that $\left\{x_j \otimes y_l \in H_1 \otimes H_2 \,\middle|\, j \in \{1, ..., k\}; l \in \{1, ..., m\}\right\}$ is linear independent.

Consider a linear combination $0 = \sum_{i=1}^{k} \sum_{j=1}^{m} \alpha_{jl}(x_j \otimes y_l)$ and $x \in H_1$, $y \in H_2$. Then

$$\begin{aligned} 0 &= \left(\sum_{j=1}^{k} \sum_{l=1}^{m} \alpha_{jl}(x_j \otimes y_l) \right)(x, y) = \sum_{j,1}^{k} \sum_{l=1}^{m} \alpha_{jl} \left\langle x_j, x \right\rangle_{H_1} \left\langle y_l, y \right\rangle_{H_2} \\ &= \left\langle \sum_{j=1}^{k} \sum_{l=1}^{m} \alpha_{jl} \left\langle y_l, y \right\rangle_{H_2} x_j, x \right\rangle. \end{aligned}$$

This holds for all $x \in H_1$. Thus we get $\sum_{i,j=1}^{k,m} \alpha_{ij} \left\langle y_l, y \right\rangle_{H_2} x_j = 0$. Since $\left\{x_j \,\middle|\, j = 1, ..., k\right\}$ is linear independent, we get

$$0 = \sum_{l=1}^{m} \alpha_{jl} \left\langle y_l, y \right\rangle = \left\langle \sum_{l=1}^{m} \alpha_{jl} y_l, y \right\rangle \quad \text{for } j = 1, ..., k \text{ and all } y \in H_2.$$

Again we get $\sum_{l=1}^{m} \alpha_{jl} y_l = 0$ for $j \in \{1, ..., k\}$ and by linear independence $\alpha_{jl} = 0$ for all $j \in \{1, ..., k\}$ and $l \in \{1, ..., m\}$.

This concludes the proof. □

Definition A.5 (Tensorproduct of linear operators)**.** Let A, B be linear operators and $A \overset{a}{\otimes} B$ be closable. Then define $A \otimes B$ as the closure of $A \overset{a}{\otimes} B$.

As we will see in the following Lemma A.7 the existence proof of the closure depends on the adjoint operators. For the Definition 2.4 of adjoint operators we needed that the operator is densely defined. For the algebraic tensor product of two densely defined operators this is the case as we have to show.

Lemma A.6 (Dense subsets)**.** *Let $S_1 \subseteq H_1$ and $S_2 \subseteq H_2$ such that $\operatorname{Lin} S_i$ is dense in H_i ($i \in \{1, 2\}$). Then $S_1 \overset{a}{\otimes} S_2$ is dense in $H_1 \otimes H_2$.*

Proof. Firstly consider a pure tensor $x \otimes y \in H_1 \otimes H_2$ that is to be approximated by elements of $S_1 \overset{a}{\otimes} S_2$.

There exist sequences $(x_n)_{n\in\mathbb{N}}$ in $\operatorname{Lin} S_1$ and $(y_n)_{y\in\mathbb{N}}$ in $\operatorname{Lin} S_2$, such that $x_n \to x$ in H_1 and $y_n \to y$ in H_2 as $n \to \infty$. Then $x_n \otimes y_n \in \operatorname{Lin} S_1 \overset{a}{\otimes} \operatorname{Lin} S_2 = S_1 \overset{a}{\otimes} S_2$. For the approximation of $x \otimes y$ by $x_n \otimes y_n$ reduce it to approximation in H_1 and H_2:

$$\begin{aligned}
\|x \otimes y - x_n \otimes y_n\|_{H_1\otimes H_2} &\le \|x_n \otimes y_n - x \otimes y_n\| + \|x \otimes y_n - x \otimes y\|_{H_1\otimes H_2} \qquad \text{Triangle ineq.}\\
&= \|(x_n - x) \otimes y_n\|_{H_1\otimes H_2} + \|x \otimes (y_n - y)\|_{H_1\otimes H_2}\\
&\overset{(31)}{=} \|x_n - x\|_{H_1} \|y_n\|_{H_2} + \|x\|_{H_1} \|y_n - y\|_{H_2}\\
&\to 0 \cdot \|y\|_{H_2} + \|x\|_{H_1} \cdot 0 \to 0 \quad (n \to \infty)
\end{aligned}$$

Linear combinations of pure tensors (that is, elements of $H_1 \overset{a}{\otimes} H_2$), can be approximated by approximating the summands. Since $H_1 \overset{a}{\otimes} H_2$ is dense in $H_1 \otimes H_2$ by Definition A.3, $S_1 \overset{a}{\otimes} S_2$ is dense in $H_1 \otimes H_2$. □

Lemma A.7. *Let $A\colon H_1 \supseteq \mathrm{D}(A) \to K_1$, $B\colon H_2 \supseteq \mathrm{D}(B) \to K_2$ be densely defined, closable, linear operators. Then $A \overset{a}{\otimes} B$ is closable and*

$$A \otimes B = \overline{A \overset{a}{\otimes} B} \subseteq (A^* \overset{a}{\otimes} B^*)^* \tag{34}$$

Proof. It is enough to show 34 since a closed subset of the (graph of the) closed operator $(A^* \overset{a}{\otimes} B^*)^*$ must be a closed operator. A^* and B^* are densely defined by Lemma 2.8 since A and B are closable. Hence $A^* \overset{a}{\otimes} B^*$ is densely defined by Lemma A.6 and thus $(A^* \overset{a}{\otimes} B^*)^*$ is a well-defined, closed, linear operator by Lemma 2.6. Since $(A^* \overset{a}{\otimes} B^*)^*$ is closed, it is enough to check, that every element η of $\mathrm{D}(A \overset{a}{\otimes} B)$ satisfies

$$\left\langle (A^* \overset{a}{\otimes} B^*)\xi, \eta \right\rangle_{H_1\otimes H_2} = \left\langle \xi, (A \overset{a}{\otimes} B)\eta \right\rangle_{K_1\otimes K_2} \quad \text{for all } \xi \in \mathrm{D}(A^* \overset{a}{\otimes} B^*)$$

since this implies $\eta \in \mathrm{D}((A^* \overset{a}{\otimes} B^*)^*)$ and $(A^* \overset{a}{\otimes} B^*)^*\eta = (A \overset{a}{\otimes} B)\eta$. Write

$$\eta = \sum_{j=1}^{m} u_j \otimes v_j \quad (m \in \mathbb{N},\, u_j \in \mathrm{D}(A),\, v_j \in \mathrm{D}(B),\, j \in \{1, ..., m\})$$

$$\text{and } \xi = \sum_{i=1}^{n} x_i \otimes y_i \quad (n \in \mathbb{N},\, x_i \in \mathrm{D}(A^*),\, y_i \in \mathrm{D}(B^*),\, i \in \{1, ..., n\}).$$

Then

$$\left\langle (A^* \overset{a}{\otimes} B^*)\xi, \eta \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \left\langle A^* x_i, u_j \right\rangle_{K_1} \left\langle B^* y_i, v_j \right\rangle_{K_2}$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \left\langle x_i, A u_j \right\rangle_{K_1} \left\langle y_i, B v_j \right\rangle_{K_2}$$
$$= \left\langle \xi, (A \overset{a}{\otimes} B)\eta \right\rangle_{K_1 \otimes K_2}. \qquad \square$$

Example A.8. The only case of operators on tensor products we will see are those of the form $A \otimes \mathrm{id}\colon H_1 \otimes H_2 \supset \mathrm{D}(A) \otimes H_2$ with id being the identity on H_2. In those cases we will write A both to denote $A\colon H_1 \supset \mathrm{D}(A)$ as well as $A \otimes \mathrm{id}$. In the case of bounded A we have for the operator norm $\|A\|_{L(H_1,H_1)} = \|A \otimes \mathrm{id}\|_{L(H_1 \otimes H_2, H_1 \otimes H_2)}$ as one can easily check for pure tensors.

A.2. Hilbert space valued L_2-space

The intention of introducing tensor products was to generalise the solution theory from scalar differential equations to systems of differential equations and even further to any Hilbert space valued functions, not just $\mathbb{C}^n$. Now we will show that tensor products are indeed the suitable tool to consider those systems.

When the ranges H are finite-dimensional it can be easily seen, that $L_2(\mathbb{R}) \otimes H \cong L_2(\mathbb{R}; H)$ where here the last is the space of square-integrable vector-valued functions: every $f \in L_2(\mathbb{R}; H)$ can be written as a vector of $L_2(\mathbb{R})$ functions hence $L_2(\mathbb{R}; H)$ is a direct sum of $\dim(H)$ copies of $L_2(\mathbb{R})$. In the infinite-dimensional case this idea still works out but needs more justification.

Theorem A.9. *Let H be a complex Hilbert space and (Ω, Σ, μ) a measure space. Then $L_2(\Omega, \Sigma, \mu; \mathbb{C}) \otimes H$ is isometrically isomorphic to $L_2(\Omega, \Sigma, \mu; H)$, the space of square-integrable H-valued functions via the identification*

$$f \otimes \phi \mapsto (x \mapsto f(x)\phi) \tag{35}$$

Proof. Let $L = L_2(\Omega, \Sigma, \mu; \mathbb{C})$ and $K = L_2(\Omega, \Sigma, \mu; H)$.

The idea is the same as for the finite dimensional case: given an $s \in K$ and a basis of H, one can determine for every ϕ of the basis and every $x \in \Omega$ the portion of ϕ in $s(x) \in H$: $\langle \phi, s(x) \rangle_H$. For all $x \in \Omega$ together this gives a function $f_\phi \in L$. By the given identification s is identified with a pure tensor.

Let U be a mapping to K defined for all pairs $f \in L$, $\phi \in H$ by the identification (35). U is isometric:

$$\|f \otimes \phi\|_{L \otimes H}^2 = \langle f, f \rangle_L \langle \phi, \phi \rangle_H = \|\phi\|_H^2 \int_\Omega |f|^2 \, d\mu$$
$$= \int_\Omega |f|^2 \|\phi\|_H^2 \, d\mu = \int_\Omega \|f(x)\phi\|_H^2 \, \mu(dx) = \|U(f \otimes \phi)\|_K^2 .$$

We can extend U linearly to $L \overset{a}{\otimes} H$. We have to show that it is a well-defined mapping. Let $0 = \sum_{i=1}^n f_i \otimes \phi_i \in L \overset{a}{\otimes} H$. (By (30) this is covers all linear combinations of pure tensors.) We have to show, that $\sum_{i=1}^n f_i \phi_i = 0$. Consider

$$\begin{aligned}
\left\langle \sum_{i=1}^n f_i \phi_i, \sum_{i=1}^n f_j \phi_j \right\rangle_K &= \sum_{i=1}^n \sum_{j=1}^n \int_\Omega \left\langle f_i \phi_i, f_j \phi_j \right\rangle_H d\mu \\
&= \sum_{i=1}^n \sum_{j=1}^n \int_\Omega f_i f_j^* \left\langle \phi_i, \phi_j \right\rangle_H = \sum_{i=1}^n \sum_{j=1}^n \left\langle f_i, g \right\rangle_L \left\langle \phi_i, \psi \right\rangle_H \\
&= \left\langle \sum_{i=1}^n f_i \otimes \phi_i, \sum_{j=1}^n f_j \otimes \phi_j \right\rangle_{L \otimes H} = \left\langle 0, 0 \right\rangle_{L \otimes H} = 0.
\end{aligned}$$

Since $\langle \cdot, \cdot \rangle_K$ is positive definite, $\sum_{i=1}^n f_i \phi_i = 0$. By linearity of U, U is well-defined.

Next, we show that that U is an isometry on $L \overset{a}{\otimes} H$. Consider $f = \sum_{i=1}^n f_i \otimes \phi_i \in L \overset{a}{\otimes} H$. Then

$$\begin{aligned}
\left\| \sum_{i=1}^n f_i \otimes \phi_i \right\|_{L \otimes H}^2 &= \sum_{i=1}^n \sum_{j=1}^n \left\langle f_i, f_j \right\rangle_L \left\langle \phi_i, \phi_j \right\rangle_H \\
&= \sum_{i=1}^n \sum_{j=1}^n \int_\Omega f_i f_j^* \, d\mu \, \left\langle \phi_i, \phi_j \right\rangle_H \\
&= \sum_{i=1}^n \sum_{j=1}^n \int_\Omega \left\langle f_i(x) \phi_i, f_j(x) \phi_j \right\rangle_H \mu(dx) \\
&= \int_\Omega \sum_{i=1}^n \sum_{j=1}^n \left\langle f_i(x) \phi_i, f_j(x) \phi_j \right\rangle_H \mu(dx) \\
&= \int_\Omega \left\| \sum_{i=1}^n f_i(x) \phi_i \right\|_H^2 \mu(dx) = \left\| \sum_{i=1}^n f_i \phi_i \right\|_K^2 = \left\| U \left(\sum_{i=1}^n f_i \otimes \phi_i \right) \right\|_K^2 .
\end{aligned}$$

Since U is an isometry on $L \overset{a}{\otimes} H$ it is in particular continuous on a dense subset of $L \overset{a}{\otimes} H$ and can by Lemma 2.3 be extended to the completion $L \otimes H$ as an isometry. It remains to show that U is surjective. For this we need to show that the range of U on $L \overset{a}{\otimes} H$ is dense in K:

Lemma A.10. *$U(L \overset{a}{\otimes} H)$ is dense in K.*

Proof. Let S be a orthonormal basis of H, possibly non-countable. Let $t \in K$. Let $t_s \in L$ with $t_s(x) = \langle t(x), s \rangle_H$ be the portion of t in the direction s. Let

$$S_n = \left\{ s \in S \,\middle|\, \|t_s\|_L > \frac{1}{n} \right\}.$$

Let $S' \subseteq S_n$ be finite. Then by Bessel's inequality, see [V.4.3 Wer11, pp. 233 sq.]

$$\begin{aligned}\|t\|_K^2 &= \int_\Omega \|t(x)\|_H^2 \, d\mu \geq \int_\Omega \sum_{s \in S'} |\langle t(x), s\rangle|^2 \, d\mu \\ &= \sum_{s \in S'} \int_\Omega |\langle t(x), s\rangle|^2 \, d\mu = \sum_{s \in S'} \|t_s\|_L^2 > \frac{1}{n^2} \sum_{s \in S'} 1\end{aligned}$$

If any S_n was infinite, $\|t\|_K^2$ would be unbounded, a contradiction. Hence

$$\tilde{S} = \bigcup_{n \in \mathbb{N}} S_n = \{s \in S \,|\, t_s \neq 0\} = \{s_1, ...\}$$

is countable. By Parseval's identity in H, see [Satz V.4.9 Wer11, p. 236]

$$\begin{aligned}\lim_{k \to \infty} \left\| t - \sum_{i=1}^{k} t_{s_i} s_i \right\|_K^2 &= \lim_{k \to \infty} \int_\Omega \left\| t - \sum_{i \in \mathbb{N}}^{k} t_{s_i} s_i \right\|_H^2 \, d\mu \\ &= \int_\Omega \lim_{k \to \infty} \left\| t - \sum_{i \in \mathbb{N}}^{k} t_{s_i} s_i \right\|_H^2 \, d\mu = 0\end{aligned}$$

where the exchange of limit and integral is justified by the monotone convergence theorem together with Parseval's identity. Hence t can be approximated by elements of $U(L \otimes H)$, i. e. $U(L \otimes H)$ is dense in K. □

□

B. Density of $C_c^\infty(\mathbb{R})$ in $L_2(\mathbb{R})$

Elements of the common Banach and Hilbert spaces like $L_2(\mathbb{R})$ are functions with a number of properties that makes them very unhandy to work with: they are only defined as equivalence classes, they are not differentiable or even continuous and not closed under multiplication. So often one can only write down explicit formulas for some special functions like differentiable ones. But then it is often possible to extend the results to all of $L_2(\mathbb{R})$ by the tools that the functional analysis provides via taking limits. But in order to extend any results we have to know that we can approximate functions in $L_2(\mathbb{R})$ via nice functions. The "nice" functions that are usually used are smooth functions with compact support.

Definition B.1 (Testfunctions)**.** Define for any Hilbert space H

$$C_c^\infty(\mathbb{R}; H) := \left\{ f\colon \mathbb{R} \to H \;\middle|\; \begin{array}{c} \operatorname{supp}(f) \text{ is compact and} \\ f^{(n)} \text{ is continuously differentiable for every } n \in \mathbb{N} \end{array} \right\}.$$

If the range is not given, we take $H = \mathbb{C}$: $C_c^\infty(\mathbb{R}) = C_c^\infty(\mathbb{R}, \mathbb{C})$.

To investigate an arbitrary $f \in L_2(\mathbb{R})$, we approximate f with smooth functions with bounded support. To be able to do that we truncate f and "make it smooth" with an operation called convolution. For this procedure we need the following family of functions:

Definition B.2 (Friedrichs mollifier)**.** A sequence $(\delta_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R})$ of smooth functions is called a *Friedrichs mollifier* or *δ-sequence* if the following holds for all $n \in \mathbb{N}$:

$$\delta_n \geq 0, \tag{36}$$

$$\operatorname{supp}(\delta_n) \subseteq \left[-\frac{1}{n}, \frac{1}{n}\right], \tag{37}$$

$$\int_{\mathbb{R}} \delta_n = 1. \tag{38}$$

So the δ_n are bumps around 0 that get steeper while n grows. One might think of them as approximations of a "function" that is zero on $\mathbb{R} \setminus \{0\}$ and such big infinity at $\{0\}$ that the integral is 1 even though there is of course no such function.

The existence of such a sequence, even the existence of any elements of $C_c^\infty(\mathbb{R})$ is not completely obvious. That is why we give one example:

Example B.3 (Friedrichs mollifier). Let $\hat{\delta}\colon \mathbb{R} \to \mathbb{C}$, $x \mapsto \begin{cases} \exp(-\frac{1}{1-x^2}) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$.

Apparently $\operatorname{supp}(\hat{\delta}) \subseteq [-1, 1]$. To show that all derivatives exist, note that on $(-1, 1)$ $\hat{\delta}$ is smooth as a composition of smooth functions. We have to check that $\lim_{x\to\pm 1} \hat{\delta}^{(k)}(x) = 0$ for every $k \in \mathbb{N}_0$. Consider any derivative of $\exp(-\frac{1}{1-x^2})$. It is of the form

$$\exp\left(-\frac{1}{1-x^2}\right) \frac{P(x)}{(1-x^2)^n} \text{ with a polynomial } P.$$

For $x \to \pm 1$ the exponential term dominates over any polynomial and hence

$$\lim_{x \to \pm 1} \delta^{(k)}(x) = 0.$$

$\hat{\delta}$ does not satisfy the integration conditions. Therefore consider

$$\delta := \left(\int_{\mathbb{R}} \hat{\delta}(x)\, dx \right)^{-1} \hat{\delta},$$
$$\delta_n : \mathbb{R} \to \mathbb{C}, x \mapsto n\delta(nx) \qquad (n \in \mathbb{N}).$$

Definition B.4 (Convolution)**.** Let $f, g \in L_2(\mathbb{R})$. Then $f * g \colon \mathbb{R} \to \mathbb{C}$ is called the *convolution* of f and g and is defined by

$$f * g \colon x \mapsto \int_{\mathbb{R}} f(x-y) \cdot g(y) dy \overset{z=x-y}{=} \int_{\mathbb{R}} f(z) \cdot g(x-z)\, dz$$

Lemma B.5. *The convolution is well-defined and commutative.*

Proof. With the notation from the definition, we see, that for every $x \in \mathbb{R}$, $x \mapsto f(x-y)$ is in $L_2(\mathbb{R})$ and therefore $\langle x \mapsto f(x-y), g \rangle = (f * g)(x) \in \mathbb{R}$. With the formula in the definition, we see that $f * g = g * f$. ☐

As noted after the Definition B.2, one might think of

$$\lim_{n \to \infty} \delta_n \text{ as } \infty \chi_{\{0\}} \text{ with } \int_{\mathbb{R}} \lim_{n \to \infty} \delta_n(x)\, dx = 1.$$

With that in mind, it sounds reasonable to hope that this calculation holds in some way:

$$\begin{aligned} f * \left(\lim_{n \to \infty} \delta_n \right)(x) &= \int f(z) \left(\lim_{n \to \infty} \delta_n \right)(x-z)\, dz \\ &= \int \left(\begin{cases} 0 & z \neq x \\ f(x)(\lim_{n \to \infty} \delta_n)(x) & z = x \end{cases} \right) dz = f(x). \end{aligned}$$

Of course such a function does not exist and therefore this not correct but it nevertheless captures the idea to approximate f by convoluting it with a Friedrichs mollifier. (To make this precise one needs distribution theory which we will not go into detail here.) This is helpful since $f * \delta_n$ is smooth.

Theorem B.6 (Convolution smoothes)**.** *Let* $f \in L_2(\mathbb{R})$, $g \in C_c^\infty(\mathbb{R})$. *Then* $f * g \in C^\infty(\mathbb{R})$ *and* $(f * g)^{(n)} = (f * g^{(n)})$ *for every* $n \in \mathbb{N}$.

Proof. Let $f \in L_2(\mathbb{R})$, $g \in C_c^\infty(\mathbb{R})$. Then $(f * g)(x) = \int_{\mathbb{R}} f(z) \cdot g(x-z)\, dz$. Let $x, h \in \mathbb{R}$, $|h| < 1$. Then

$$\begin{aligned} \frac{(f * g)(x+h) - (f * g)(x)}{h} &= \frac{\int_{\mathbb{R}} f(z) g(x+h-z)\, dz - \int_{\mathbb{R}} f(z) g(x-z)\, dz}{h} \\ &= \int_{\mathbb{R}} f(z) \frac{g(x+h-z) - g(x-z)}{h}\, dz\,. \end{aligned}$$

The function g is continuously differentiable, hence $\frac{g(x+h-z)-g(x-z)}{h} \to g'(x-z)$ for every $z \in \mathbb{R}$. To apply the Lebesgue convergence theorem, there needs to exist a majorizing function. By the mean value theorem there exists for every $h, z \in \mathbb{R}$ an $\zeta_{h,z}$ between $x+h-z$ and $x-z$ with $\frac{g(x+h-z)-g(x-z)}{h} = g'(\zeta_{h,z})$. The function g' is bounded since it is continuous on a compact domain, hence

$$\begin{aligned} \left| f(z)\frac{g(x+h-z)-g(x-z)}{h} \right| &\leq \left| \chi_{\operatorname{supp}(g)+[-1,1]}(z) f(z) \right| \sup_{y\in\mathbb{R}} |g'(y)| \\ \text{and } \left| \chi_{\operatorname{supp}(g)+[-1,1]} f \right| \sup_{y\in\mathbb{R}} |g'(y)| \in L^1(\mathbb{R}) \end{aligned}$$

since for a finite measure space Ω (here $\Omega = \operatorname{supp}(g) + [-1, 1]$), it holds that $L_2(\Omega)$ is a subspace of $L_1(\Omega)$. By the Lebesgue convergence theorem,

$$\frac{(f*g)(x+h)-(f*g)(x)}{h} \to \int_{\mathbb{R}} f(z)g'(x-z)\,dz = (f*g')(x).$$

By induction, the statement follows. ☐

Remark. With the argumentation of the last proof, it also follows that the convolution of two functions of which the smoother one has compact support is (at least) as smooth as the smoother one.

Lemma B.7 (Support of the convolution)**.** *Let* $f, g \in L_2(\mathbb{R})$, *then* $\operatorname{supp}(f*g) \subseteq \operatorname{supp}(f) + \operatorname{supp}(g) := \{x + y \mid x \in \operatorname{supp}(f),\, y \in \operatorname{supp}(g)\}$.

Proof. Let $x \in \mathbb{R}$. If there is any $z \in \mathbb{R}$ with $f(z)g(x-z) \neq 0$, then

$$\begin{aligned} f(z)g(x-z) \neq 0 &\implies z \in \operatorname{supp} f \wedge x - z \in \operatorname{supp} g \\ &\iff z \in \operatorname{supp} f \wedge x \in z + \operatorname{supp} g \\ &\implies x \in \operatorname{supp} f + \operatorname{supp} g. \end{aligned}$$

Therefore $\overline{\operatorname{supp}(f*g)} \subseteq \overline{\operatorname{supp} f + \operatorname{supp} g} = \operatorname{supp} f + \operatorname{supp} g$. ☐

To finish the proof of $C_c^\infty(\mathbb{R})$ being dense in $L_2(\mathbb{R})$ we need one non-trivial fact which will not be proven here and can be read in [1.26(2) Alt12, p. 64]. At [2.15 Alt12, pp. 115 sqq.] one can also find a detailed proof and further reading material on dense subsets of L_p.

Theorem B.8 (Continuous compactly supported functions dense in L_2)**.** $C_c^0(\mathbb{R})$, *the subspace of continuous functions with compact support, is dense in* $L^p(\mathbb{R})$ *with* $1 \leq p < \infty$ *with respect to the* $\|\cdot\|_{L^p}$*-norm.*

Theorem B.9 (Approximation via convolution)**.** *Let* $(\delta_n)_{n\in\mathbb{N}}$ *be a Friedrichs mollifier,* $f \in L_2(\mathbb{R})$. *Then* $f * \delta_n \in L_2(\mathbb{R})$ *for every* $n \in \mathbb{N}$ *and*

$$\lim_{n\to\infty} \|f*\delta_n - f\|_{L_2(\mathbb{R})} = 0.$$

Proof. Let $(\delta_n)_{n\in\mathbb{N}}$, f as in the theorem. Then

$$\begin{aligned}
\|f*\delta_n - f\|^2_{L_2(\mathbb{R})} &= \int_{\mathbb{R}} |(f*\delta_n)(x) - f(x)|^2 \, dx \\
&= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(z)\cdot\delta_n(x-z)\,dz - f(x)\cdot 1 \right|^2 dx \\
&\overset{(38)}{=} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(z)\cdot\delta_n(x-z)\,dz - f(x)\cdot \int_{\mathbb{R}} \delta_n(x-z)\,dz \right|^2 dx \\
&= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (f(z)-f(x))\cdot\delta_n(x-z)\,dz \right|^2 dx \\
&\le \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(z)-f(x)|\cdot\delta_n(x-z)\,dz \right]^2 dx
\end{aligned}$$

because of the triangle inequality. The Cauchy-Schwarz inequality for

$$\text{for } a = |f(\cdot)-f(x)|\cdot\delta_n(x-\cdot)^{\frac{1}{2}} \text{ and } b = \delta_n(x-\cdot)^{\frac{1}{2}}$$

yields the following. a and b are well-defined, since (36) $\delta_n \ge 0$.

$$\begin{aligned}
\|f*\delta_n - f\|^2_{L_2(\mathbb{R})} &\le \int_{\mathbb{R}} \left[\left(\int_{\mathbb{R}} \left[|f(z)-f(x)|\cdot\delta_n(x-z)^{\frac{1}{2}} \right]^2 dz \right)^{\frac{1}{2}} \right. \\
&\qquad \left. \left(\int_{\mathbb{R}} \left(\delta_n(x-z)^{\frac{1}{2}} \right)^2 dz \right)^{\frac{1}{2}} \right]^2 dx \\
&\overset{(38),(36)}{=} \int_{\mathbb{R}} \int_{\mathbb{R}} |f(z)-f(x)|^2 \cdot \delta_n(x-z)\,dz \cdot 1^2\,dx
\end{aligned}$$

With the substitution $t = z - x$ we get

$$\begin{aligned}
\|f*\delta_n - f\|^2_{L_2(\mathbb{R})} &\le \int_{\mathbb{R}} \int_{\mathbb{R}} |f(t+x)-f(x)|^2 \cdot \delta_n(-t)\,dt\;dx \\
&\overset{\text{Fubini}}{=} \int_{\mathbb{R}} \delta_n(-t) \int_{\mathbb{R}} |f(t+x)-f(x)|^2\,dx\;dt \\
&\overset{(37)}{=} \int_{[-\frac{1}{n},\frac{1}{n}]} \delta_n(-t)\, \|f(t+\cdot)-f\|^2_{L_2(\mathbb{R})}\,dt\,.
\end{aligned}$$

The use of the Fubini's Theorem is justified because the last integral is finite which will be shown.

Now Theorem B.8 implies that there exists for any $\varepsilon > 0$ an $f_\varepsilon \in C^0_c(\mathbb{R})$ such that $\|f_\varepsilon - f\|_{L_2(\mathbb{R})} < \frac{\varepsilon}{3}$. Since f_ε has compact support $|\text{supp}(f_\varepsilon(t+\cdot)-f_\varepsilon)|$ is bounded for small t and f_ε is uniformly continuous. Hence, we can find a $t \in \mathbb{R}\setminus\{0\}$ such that

$|\mathrm{supp}(f_\varepsilon(t+\cdot)-f_k)|^{\frac{1}{2}} \left\|f_\varepsilon(t+\cdot)-f_\varepsilon\right\|_\infty < \frac{\varepsilon}{3}$. Then

$$\begin{aligned}\|f(t+\cdot)-f\|_{L_2(\mathbb{R})} &\le \|f(t+\cdot)-f_k(t+\cdot)\|_{L_2(\mathbb{R})} + \|f_k(t+\cdot)-f_k\|_{L_2(\mathbb{R})} + \|f_k-f\|_{L_2(\mathbb{R})} \\ &\le \underbrace{2\|f_k-f\|_{L_2(\mathbb{R})}}_{\le\frac{2\varepsilon}{3}} + \underbrace{|\mathrm{supp}(f_k(t+\cdot)-f_k)|^{\frac{1}{2}}\|f_k(t+\cdot)-f_k\|_\infty}_{\le\frac{\varepsilon}{3}} \le \varepsilon.\end{aligned}$$

Hence $\|f(t+\cdot)-f\|_{L_2(\mathbb{R})} \to 0$ as $t\to\infty$. Then we arrive at

$$\begin{aligned}\|f*\delta_n-f\|^2_{L_2(\mathbb{R})} &\le \int_{[-\frac{1}{n},\frac{1}{n}]} \delta_n(-t)\,\|f(t+\cdot)-f\|^2_{L_2(\mathbb{R})}\,dt \\ &\le \int_{[-\frac{1}{n},\frac{1}{n}]} \delta_n(-t)\,dt \sup_{|t|\le\frac{1}{n}} \|f(t+\cdot)-f\|^2_{L_2(\mathbb{R})} \\ &\to 1\cdot 0,\ n\to\infty\end{aligned}$$

□

Theorem B.10 (Test functions dense in L_2)**.** *$C_c^\infty(\mathbb{R})$ is dense in $L_2(\mathbb{R})$.*

Proof. Let $f\in L_2(\mathbb{R})$. We have shown that f can be approximated by $f*\delta_n$ with a δ-sequence (δ_n) but $f*\delta_n$ in general has no compact support. Therefore truncate f: Let $f_k := \chi_{[-k,k]}f$ for $k\in\mathbb{N}$. Then for all $x\in\mathbb{R}$, $\left(\chi_{\mathbb{R}\setminus[-k,k]}|f|^2\right)(x)\searrow 0$ and therefore by monotone convergence with

$$\begin{aligned}\infty > \int_{\mathbb{R}}|f|^2 &\ge \int_{\mathbb{R}\setminus[-k,k]}|f|^2 \ge 0 \\ \implies \|f-f_k\|_{L_2(\mathbb{R})} &= \int_{\mathbb{R}\setminus[-k,k]}|f|^2 \to 0, k\to\infty\end{aligned}$$

That means, that $f_k\to f$ in $L_2(\mathbb{R})$. With Theorem B.6 and B.7 $f_k*\delta_n\in C_c^\infty(\mathbb{R})$ for all $k,n\in\mathbb{N}$ and $f_k*\delta_n \overset{n\to\infty}{\to} f_k \overset{k\to\infty}{\to} f$ in $L_2(\mathbb{R})$. That is, what was to be shown. □

Erklärung

Hiermit erkläre ich, dass ich die am 21. Januar 2019 eingereichte Bachelorarbeit zum Thema *Delay Gleichungen als abstrakte Differentialgleichungen in gewichteten Hilberträumen* unter Betreuung von Dr. rer. nat. habil. Sascha Trostorff selbstständig erarbeitet, verfasst und Zitate kenntlich gemacht habe. Andere als die angegebenen Hilfsmittel wurden von mir nicht benutzt.

Dresden, 21. Januar 2019 Unterschrift

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