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Orientability in Nematic Liquid Crystal Models on Surfaces

Master Thesis

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Abstract

The orientation of liquid crystal particles can be modelled with a unit vector field (Frank–Oseen model) or a line field (constrained Landau–de Gennes model). Ball and Zarnescu (2011) gave a criterion to decide if a line field on a Euclidean 2-dimensional domain is orientable. A line field is called orientable if one of the two directions of the lines in every point can be chosen without losing regularity. This thesis transfers those results to manifolds. First, it is shown that all Sobolev $W^{1,q}$ ($q \geq 2$) line fields on simply-connected manifolds of any dimension are orientable. Secondly, it is shown that a Sobolev $W^{1,2}$ line field on an orientable surface is orientable if and only if it is orientable on a set of loops that generate the fundamental group. Furthermore, an analytical formulation of orientability on a loop on an orientable surface is given. As necessary tools, different definitions of Sobolev vector and tensor fields are introduced and shown to be equivalent.

Zusammenfassung

Die Ausrichtung von Flüssigkristallteilchen kann sowohl mit einem Einheitsvektorfeld (Frank–Oseen-Modell) als auch mit einem Linienfeld (beschränktes Landau–de Gennes-Modell) beschrieben werden. Ball und Zarnescu (2011) bewiesen ein Kriterium, um zu entscheiden, ob ein Linienfeld auf einem Euklidischen 2-dimensionalen Gebiet orientierbar ist. Ein Linienfeld heißt orientierbar, wenn eine der beiden Richtungen der Linien in jedem Punkt ausgewählt werden kann, ohne Regularität zu verlieren. Diese Masterarbeit überträgt diese Resultate auf Mannigfaltigkeiten. Es wird gezeigt, dass $W^{1,q}$-Sobolevlinienfelder ($q \geq 2$) auf einfach zusammenhängenden Mannigfaltigkeiten jeder Dimension orientierbar sind. Zweitens ist ein $W^{1,2}$-Sobolevlinienfeld auf einer orientierbaren Oberfläche genau dann orientierbar, wenn es auf einer Menge von Schlaufen, welche die Fundamentalgruppe erzeugen, orientierbar ist. Außerdem wird Orientierbarkeit auf einer Schlaufe auf einer orientierbaren Mannigfaltigkeit auch analytisch formuliert. Verschiedene Definitionen von Sobolevvektor- und -tensorfeldern auf Mannigfaltigkeiten werden als benötigte Werkzeuge eingeführt und ihre Äquivalenz wird gezeigt.

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Feedback is welcome. The author can be reached via `felix.h@hilsky.de`.

Contents

List of theorems, definitions, examples and remarks

1 Introduction

1.1 Physical background of nematic liquid crystals

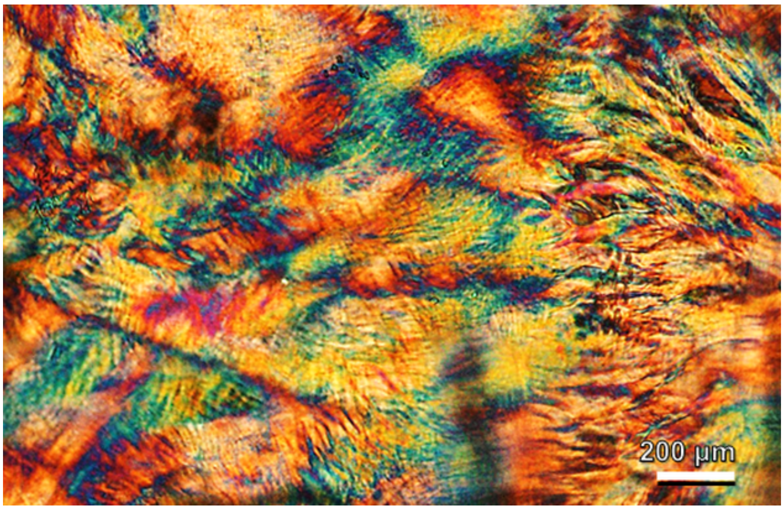


Figure 1: A microscopic image of a liquid crystal. We see the effect of double refraction. The colors are caused by the local order of direction which changes with location and time. Polarized light is shining through this plate while the camera has a perpendicular polarization filter such that we only see colours where the liquid crystal changes the polarization of the light. Image from Joshi et al. [Jos+19].

In 1888, the botanist Friedrich Reinitzer heated cholesterol extracted from carrots. At room temperature it was solid, at 145 °C it looked milky and appeared to be between solid and liquid. After further heating until 179 °C the cholesterol became a clear liquid. The physicist and crystallographer Otto Lehmann then studied this phenomenon. He explained the milkiness with double refraction in the cholesterol. [Che11, Ch. 5]

By this point double refraction was already well-known from solid crystals.

What are nematic uniaxial liquid crystals? Crystals are highly structured solid matter. Namely the molecule position and orientation is ordered. This causes different refraction indices depending on the polarity of the light. This effect can be seen in Figure 2.

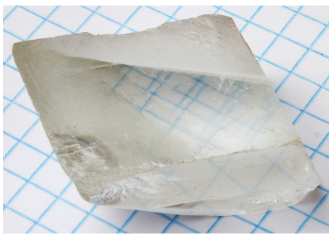


Figure 2: Double refraction of a calcite crystal. The two images of a blue line come from the two orthogonal polarities which are refracted by a different angle. Image by APN MJM [APN11].

Lehmann saw the same effect in the heated cholesterol and deduced that this liquid also had some internal structure. He therefore coined the term *flüssiger Kristall*, in

English *liquid crystal*.

In a liquid crystal, the molecules locally are roughly aligned in the same direction. In some liquid crystals, there is also a partial ordering in the position of the molecules. Otherwise, they are positioned isotropically. Liquid crystals can move like a liquid but are in many cases very viscous. [Dem+98, p. II.2]

Since Reinitzer's discovery, it was found that many substances can be in a phase between liquid and crystal. The different ways how they are structured are called *mesophases*. 'Meso' means middle or in between. The 'phase' is the state that matter can be in, usually solid, liquid or gaseous.

The *nematic* phase is the simplest and best-studied liquid crystal mesophase. The word *nematic* comes from the Greek word *nema*, meaning *thread*. This refers to the lines that are visible under the microscope as in Figure 1. In a nematic liquid crystal, the *position* of the molecules is isotropic. That means that there is no structure in it. The *orientation* of the molecules on the other hand is very similar to their neighbours. That means that their orientation is locally structured.

What are examples of liquid crystals? Most liquid crystals are polymers. Polymers are molecules that consist of many repeating, mostly carbon-based, parts. Hence it is not surprising that a lot of current liquid crystal research is in biology and biophysics.

Liquid crystals exist not only in the lab as the aforementioned carrot cholesterol but also occur naturally. Indeed, mucus on the skin of slugs is a liquid crystal. The molecules are proteins called mucin. The liquid crystal structure holds the mucus together and is responsible the slimy feeling. Since it is also able to adapt, it protects the slug against sharp edges of sand or rocks. The article [McQ16] describes how the same principle applies to giraffe saliva that protects their tongue against thorns.

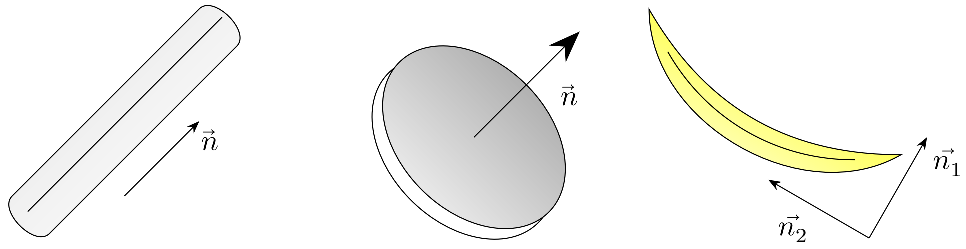
For applications in technology researchers try to understand how organic molecules like proteins or polysaccharides assemble and form hierachical structures. One example is the work of Joshi et al. [Jos+21] who study nature-derived polysaccharides. They evaporate the water from an aqueous liquid crystalline solution and the polysaccharide units self-assemble and thus create a thin membrane upon dehydration. This resulting liquid crystal is uniaxial. This differs from the cholesterol mentioned at the beginning which becomes a liquid crystal at the certain temperature. By using rods and disc-shaped liquid crystal units in combination, the created membrane is bridging 8 mm instead of 1 mm with just rod-shaped liquid crystal units.

Another instance of liquid crystals that are formed by self-assembly was found by Morales-Navarrete et al. [Mor+19] in liver tissue. These researchers were surprised to find a long-range liquid crystal order where the constituents were not molecules but cells.

We see from those very different examples that the term 'liquid crystal' describes a diverse set of substances. That makes this mesophase so interesting and worthy of study.

1.2 Models for liquid crystals

How do we model liquid crystals? The molecule orientation can be described with a single direction per point in case of *uniaxial* liquid crystals. In case of *biaxial* liquid crystals, we need two orthogonal directions as in Figure 3c. In this thesis, only uniaxial cases are considered. A direction is modelled with a vector of length 1, called the *director*. Figure 3 illustrates the meaning of the director for different shapes of molecules. In many uniaxial cases the molecules are rods that are much longer in one dimension than in the other two. Then, the director points from one end to the other end of the rod as in Figure 3a. In other cases the molecules are disc-shaped as in Figure 3b. Then, the director is orthogonal to the disc. In most liquid crystals, the molecules are not perfectly



(a) The director for rod-shaped molecules

(b) The director for disc-shaped molecules

(c) The two directors for banana-shaped molecules, examples of biaxial molecules

Figure 3: Three examples of how directors model the orientation of a molecule

aligned. This is illustrated in Figure 4. Therefore, the director at one point models not the direction of a single molecule but the average of the directions of the molecules around that point.

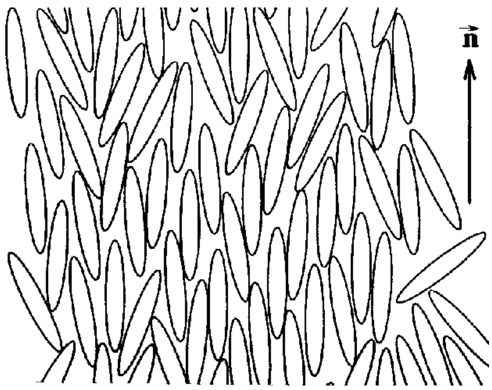


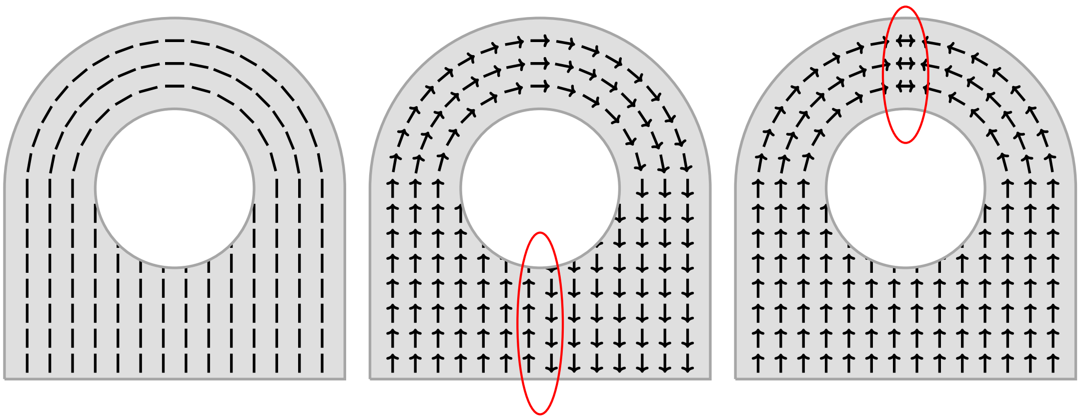
Figure 4: Average orientation of molecules in a nematic liquid crystal. From [Maj18, p. 3]

How do we model liquid crystals globally? When we take the directors in all points together, we have a unit vector field, called the *director field*. A *field* is just a vector-valued function. This is the earliest model for liquid crystals [Fra58]. It is named after Carl Wilhelm Oseen who developed it in 1933 and Frederick Charles Frank who refined it in 1958. The model further describes how to calculate how the molecules are oriented: The director field minimizes the 'Frank–Oseen energy'.

The Frank–Oseen model is simple but has shortcomings. These were addressed in 1974 by de Gennes [dGen74]. He formulated another model of so-called *Q-tensors*. It is based on the solid-liquid phase transition theory of Lev Landau [Che11, p. 162] and is therefore called Landau–de Gennes model. We address only one of the shortcomings, namely the missing head-tail symmetry. Molecules in nematic liquid crystals are head-tail symmetric, so a unit vector n models the same direction as $-n$. Therefore it is more appropriate to use a *line field* instead of a unit vector field. A line field is a special case of a Q-tensor field. Figure 5a illustrates how we can picture a line field on a flat domain. Mathematically the codomain of a line field is the real projective space.

We call a line field *orientable* if we can choose in every point one of the two unit vectors that represent the line. Additionally, the resulting unit vector field has to be continuous if the line field is continuous or weakly differentiable if the line field is weakly differentiable. Figure 5 shows an example of a continuous line field that is not orientable as explained in the caption. Since unit vector fields are easier to handle than maps into a projective space the question is:

$$\text{Is a given line field orientable?} \tag{1}$$



(a) A continuous line field.

(b) If the director is continuous above the hole, it has to turn around and is discontinuous below the hole.

(c) If the director is continuous below the hole, the vector field is discontinuous above the hole.

Figure 5: An unorientable line field on a domain with one hole and two failed attempts to find an orientation. Both are discontinuous where the line field is continuous. From [Fig. 1 BZ11]

How can we predict the behaviour of a liquid crystal? Physical models do not only describe observations but also predict how a system behaves under given parameters. In the case of liquid crystals, these parameters include the shape and topology of the domain, boundary values, electrical or magnetic fields, the temperature and pressure. They are modeled with the Frank–Oseen energy, a functional on the space of line or director fields. The orientations of the molecules then minimize this energy and thus, the mathematical formulation is a minimization problem in a class of unit vector fields or line fields.

The question that arises is:

$$\text{Is the line field energy minimizer orientable?} \tag{2}$$

If for a specific problem the answer is 'yes' and this is shown without calculating the minimizer, the minimization problem can be solved in the class of unit vector fields. An orientability criterion, i. e. an answer to Question 1, is obviously an important tool for answering this question.

Minimization problems are usually not solved in the class of continuous functions though, but in the class of weakly differentiable (Sobolev) functions. Therefore, Question 1 also needs to be answered for Sobolev line fields.

1.3 Previous results and content of this thesis

What has been shown before? Ball and Zarnescu [BZ11] showed that on simply-connected two- and three-dimensional flat domains for $q \geq 2$, all weakly differentiable $W^{1,q}$ line fields are orientable [Thm. 2 BZ11]. For two-dimensional flat domains with holes, they showed that orientability of a $W^{1,2}$ line field is equivalent to orientability on the boundaries of the holes [Prop. 7 BZ11]. Furthermore, they characterised orientability on a hole boundary with the winding number of an auxiliary unit vector valued map. Intuitively spoken, this auxiliary map turns twice as fast as the line field and if its winding number is even, the line field is orientable.

How does this thesis generalise? Some applications like [Keb+14] study thin films of liquid crystals which are best modeled as two-dimensional surfaces. Therefore, this thesis generalises the results of [BZ11] to curved surfaces. In order to stay as general as possible, we also consider manifolds of dimension higher than 2. On manifolds, the directions need to be tangent to the manifold and the tangent spaces at different points are distinct. Therefore, we need to adapt the definitions of unit vector fields, line fields and Sobolev spaces. As mentioned before, line fields are a special case of Q-tensor fields. This allows us to view line fields as tensor fields and thus, we can define what it means for a unit vector field or line fields to be tangent to the domain manifold. The Q-tensor model and the tensor notation are introduced in Section 2.

Then in Section 3, we define Sobolev spaces of tangent tensor fields. Proper discussion of Sobolev tangent tensor fields is surprisingly rare in the literature, probably because properties of Sobolev spaces for Euclidean domains can widely be transferred. Nevertheless, we need to take care because there exist several possible definitions. In Section 3.1, we define them intrinsically. The other two subsections define them differently and show that those definitions are equivalent to the intrinsic definition.

Why is the hole in the domain important? In the example in Figure 5, we already see that the hole in the domain plays a crucial role. Without it, every continuous line field is orientable. More generally, only the topology of the domain determines how we can check if a continuous line field is orientable. This follows from an algebraic topology result about liftings and will be discussed in Section 4. The idea is that the sphere in which the unit vector fields map is a double *covering* of the real projective space in which the line fields map. The theorem implies that we can check orientability of a line field by checking orientability on a set of loops that generate the fundamental group of the domain. The theorem is called the 'lifting lemma'.

How can we transfer the orientability criterion to weakly differentiable fields? The obvious way to generalise the orientability criterion from continuous to Sobolev fields is to approximate Sobolev fields by continuous fields. This is not possible, however, in the general case for manifold-valued functions. Our codomains are the sphere and the real projective space which are not vector spaces but manifolds. However, in two cases, we have sufficient density results: simply connected domains and surfaces. Surface is just a different name for a two-dimensional manifold.

How do we show the orientability criterion on simply connected domains? In the case of simply connected domains, it was shown that all Sobolev functions can be weakly approximated by smooth functions [PR03]. As explained in Note 5.3, the only way to use this theorem is to have a common codomain for all points of the domain. We combine all tangent spaces by embedding the domain into a Euclidean space. The tangent vectors are then also elements of a common Euclidean space and the Sobolev definition for $\mathbb{R}^N$-valued functions can be used. Section 3.2 discusses this definition of Sobolev functions and shows that it is equivalent with the intrinsic definition even though the norm is unequal. The smooth weak approximations that the density result give are then line fields in the surrounding space, not necessarily tangent. Fortunately the continuous orientability result holds for those as well. The last bit to the proof that all Sobolev line fields on simply-connected manifold domains are orientable (Theorem 5.4) is the stability of orientability under weak convergence (Proposition 5.2).

How do we show the orientability criterion on surfaces? The other main result considers surfaces. We show that we can check if a Sobolev $W^{1,2}$ line fields is orientable by checking it on any set of loops that generate the fundamental group of the domain. In other words, on surfaces the criterion for continuous line fields also applies to Sobolev line fields.

It turns out that all orientable surfaces either have no unit vector fields or have a trivial tangent bundle, i. e., there exists a global frame. Since line fields obviously cannot be orientable if there exist no unit vector field, we only have to consider the other case of surfaces with trivial tangent bundle. This frame allows us to identify all tangent spaces and therefore consider vector and line fields as maps into $\mathbb{R}^2$. The Sobolev function definition that this identification induces is discussed in Section 3.3, where we also show that this definition is also equivalent to the intrinsic one. The rest of the argument resembles the proof from Section 4 in [BZ11]:

The easier direction from orientability on the surface to orientability on loops is based on the trace theorem that shows that the restriction to a loop is a continuous operator between Sobolev spaces. In the other direction, we have to show that the approximations are orientable on the loops if the approximated Sobolev field is orientable on the loops. For this, we use the winding number which is an integer. The winding number says how often a circle valued function on a loop wraps around the circle and is originally defined in algebraic topology for continuous functions. It got generalised for fractional $W^{1/2,2}$ Sobolev functions, which is the suitable trace space of the line fields on surfaces [dMGP91]. To use it for line fields, we notice that on surfaces, the lines are modeled with the real projective line which is homeomorphic to the circle. As such it also has a winding number which is even exactly for orientable line fields. In contrast to higher dimensions, on surfaces, smooth unit vector valued functions are dense in the Sobolev space of unit vector valued functions with respect to the norm. Since the winding number is an integer and is continuous as a functional on the suitable Sobolev trace space, approximations that are close enough in the norm sense have the same winding number and are thus orientable on loops as well. At this point, the orientability criterion for continuous line fields is used and the limit is an orientation of the approximated line field.

What can we say about Question 2? The two orientability criteria answer the Question 1 for Sobolev line fields in special cases. As the next step, Ball and Zarnescu [BZ11] present a condition on a flat two-dimensional domain with holes for orientability of the energy minimizer of the harmonic energy $E(Q) = \int |\nabla Q| \,\mathrm{d}x$. The harmonic energy is a special case of the Frank–Oseen energy. Unfortunately, this result relies heavily on another work [BBH94] that reduces the minimization problem to a scalar problem and shows that a minimizer exists. This work specialises on flat two-dimensional domains with holes and extending it is far beyond the scope of this thesis. In Section 6, we show however that there exist a condition on the torus similar to a boundary condition for which the line field minimizer of the harmonic energy is not orientable. Therefore, the question of how to check if harmonic energy minimizers on surfaces are orientable remains open.

2 Landau–de Gennes model on manifolds

The Frank–Oseen and Landau–de Gennes model use vector-valued and matrix-valued fields when they are used on Euclidean domains. Since we will use these models on manifolds, we need to define what it means for those fields to be tangent to the manifold. The goal of this section is to introduce the Q-tensor model and how it specialises to the line fields and unit vector fields that we study in this thesis.

Throughout the thesis, let M be a smooth Riemannian manifold with or without boundary. Let g be the metric on M and $m := \dim M$.

2.1 Tensor notation

In accordance to Lee [Lee18] we denote the space of tensors relative to a vector space V that take k covectors and l vectors as arguments as $T^{(k,l)}V$. In particular a $(1,0)$-tensor is a (contravariant) vector, a $(0,1)$ tensor is a covector, a $(k,0)$-tensor is a contravariant tensor and a $(0,l)$-tensor is a covariant tensor.

In this terminology, the metric g is a $(0,2)$-tensor in every point $p \in M$.

Given a basis $(E_1, ..., E_d)$ of a vector space V, any $Q \in T^{(2,0)}V$ can be written in Einstein summation convention as

$$Q = Q^{ij} E_i \otimes E_j \ .$$

Also in accordance to [Lee18], the trace of a tensor is first defined for $(1,1)$-tensors in the same way as for vector space endomorphisms since for $t \in T^{(1,1)}V$, we get after plugging in a $v \in V$ the map $t(v, \cdot) \in V^{**} \cong V$. This shows that t is an endomorphism of V. From linear algebra, we know that $\mathrm{tr}(t)$ is then defined as the sum of the diagonal entries of any representing matrix and that this value does not depend on the choice of the basis. For $Q \in T^{(2,0)}$, we first have to lower one index. Here, $(\varepsilon^1, ..., \varepsilon^d)$ is the dual basis to $(E_1, ..., E_d)$. Then

$$\mathrm{tr}(Q) = \mathrm{tr}(Q^{ij} E_i \otimes E_j) = \mathrm{tr}(g_{ik} Q^{ij} \varepsilon^k \otimes E_j) = g_{ij} Q^{ij},$$

which is the usual trace if the chosen basis is orthonormal.

2.2 Q-tensors

The Landau–de Gennes model describes the orientation of the rod-like molecules with a probability measure μ in every point. You can also think of this measure in a point as the distribution of molecule orientations very close to this point. We write

$$\mu(p,\cdot)\colon \mathcal{L}(\mathbb{S}_pM) \to [0,1]$$

with the space of unit vectors $\mathbb{S}_pM := \{v \in T_pM \mid |v|_g = 1\}$ and the set of Lebesgue measurable subsets $\mathcal{L}(\mathbb{S}_pM)$. In every point $p \in M$, we have $\mu(p,\mathbb{S}_pM) = 1$ since $\mu(p,\cdot)$ is a probability measure. $\mu(p,S)$ gives the probability of a molecule at p pointing in a direction in $S \subseteq \mathbb{S}_pM$.

As mentioned in the introduction, the molecules are head-tail symmetric. This is modelled with the assumption that $\mu(p,-S) = \mu(p,S)$ for all $p \in M$ and measurable $S \subseteq \mathbb{S}_pM$. Here, $-S = \{-x \mid x \in S\}$. That implies that the expected value in a point $p \in M$ vanishes:

$$\int_{\mathbb{S}_pM} v \,\mathrm{d}\mu(v) = \frac{1}{2}\left(\int_{\mathbb{S}_pM} v \,\mathrm{d}\mu(v) + \int_{\mathbb{S}_pM} -v \,\mathrm{d}\mu(-v)\right) = 0 \ .$$

To describe a probability distribution, the next term to consider is the tensor of second moments:

$$\mu_2(p) := \int_{\mathbb{S}_pM} v \otimes v \,\mathrm{d}\mu(v) \ .$$

From this definition it is clear that μ_2 is symmetric.

For calculating the trace of $\mu_2(p)$, choose an orthonormal basis $(E_1,...,E_d)$ of T_pM. Then

$$\begin{aligned}\operatorname{tr}(\mu_2(p)) &= \operatorname{tr}((\mu_2)^{ij} E_i \otimes E_j) = \sum_{i=1}^{m} \mu_2^{ii} = \sum_{i=1}^{m} \int_{\mathbb{S}_pM} (v^i)^2 \,\mathrm{d}\mu(v) \\ &= \int_{\mathbb{S}_pM} \sum_{i=1}^{m} (v^i)^2 \,\mathrm{d}\mu(v) = \int_{\mathbb{S}_pM} 1 \,\mathrm{d}\mu(v) = 1 \ .\end{aligned}$$

In a normal liquid, the molecule orientations are isotropically distributed. Since we want to measure the alignment of the molecules, we consider the difference to the isotropic distribution. The second moment tensor μ_{iso} of the isotropic distribution is given by

$$\begin{aligned}\mu_{\text{iso}} &= \frac{1}{4\pi}\int_{\mathbb{S}_pM} v \otimes v \,\mathrm{d}A(v) = \frac{1}{m}\sum_{i=1}^{m} E_i \otimes E_i = \frac{1}{m} g^{\sharp\sharp} \\ \text{since } \frac{1}{4\pi}\int_{\mathbb{S}_pM} v^i v^j E_i \otimes E_j \,\mathrm{d}A(v) &= \frac{1}{4\pi}\int_{\{v \in \mathbb{S}_pM \mid v^i \geq 0\}} v^i v^j E_i \otimes E_j \,\mathrm{d}A(v) \\ &\quad + \frac{1}{4\pi}\int_{\{v \in \mathbb{S}_pM \mid v^i < 0\}} v^i v^j E_i \otimes E_j \,\mathrm{d}A(v) \\ &= \frac{1}{4\pi}\int_{\{v \in \mathbb{S}_pM \mid v^i \geq 0\}} (v^i v^j + (-v^i) v^j)\, E_i \otimes E_j \,\mathrm{d}A(v) \\ &= 0 \text{ for } i \neq j\end{aligned}$$

$$\text{and } \frac{1}{4\pi}\int_{\mathbb{S}_pM}(v^i)^2 E_i\otimes E_i \,\mathrm{d}A(v) = \frac{1}{4\pi}\left(\frac{1}{m}\int_{\mathbb{S}_pM}\underbrace{\sum_{k=1}^{m}(v^k)^2}_{=|v|^2=1}\mathrm{d}A(v)\right)E_i\otimes E_i$$
$$= \frac{1}{4\pi}\cdot\frac{1}{m}\left|\mathbb{S}_pM\right| E_i\otimes E_i$$
$$= \frac{1}{m}E_i\otimes E_i \;.$$

As usual $g^{\sharp\sharp} = \sum_{i=1}^m E_i\otimes E_i$ is the metric with twice raised indices, hence a $(2,0)$-tensor.

We call for any point p on M

$$Q_p := \mu_2 - \mu_{\mathrm{iso}} = \int_{\mathbb{S}_pM} v\otimes v - \frac{1}{m}g^{\sharp\sharp}\,\mathrm{d}\mu(v) \tag{3}$$

a *Q-tensor*. Q-tensors are symmetric and trace-free as shown above.

Definition 2.1 (Q-tensor bundle)**.** The tensor bundle of Q-tensors is defined as

$$\mathcal{Q}M := \left\{Q\in T^{(2,0)}TM \,\middle|\, Q^T = Q \text{ and } \operatorname{tr}Q = 0\right\}$$
$$\text{with the fiber } \mathcal{Q}_pM := \left\{Q\in T^{(2,0)}T_pM \,\middle|\, Q^T = Q \text{ and } \operatorname{tr}Q = 0\right\} \text{ at any } p\in M\;. \quad ◀$$

For a more thorough introduction to Q-tensor theory see, [MN14].

Most research considers Q-tensors in Euclidean space. Then, Q_p is a matrix without specification if it is a $(2,0)$-, $(1,1)$- or $(0,2)$-tensor. Since we consider a Riemannian manifold, the metric induces the linear isometric musical isomorphisms that raise and lower indices. Therefore both co- and contravariant tensors can be interpreted to describe the molecule alignment. The paper [NRV20] treats liquid crystals on surfaces and uses a contravariant Q-tensor field. In another paper [Nit+18], the same working group uses a covariant Q-tensor field though. Therefore, there is no agreement in the literature. Since the director field is modelling a direction, the obvious choice is a contravariant vector and thus, the obvious choice for second moment tensor is contravariant as well.

2.3 The constrained Landau–de Gennes model

In order to model line fields in the Q-tensor theory, consider the spectral representation of a symmetric trace-free $Q_p\in\mathcal{Q}_pM$ ($p\in M$)

$$Q_p = \lambda^i E_i\otimes E_i \text{ with } \lambda_m = -\sum_{i=1}^{m-1}\lambda^i \text{ and } E_i \text{ an orthonormal basis.}$$

If the eigenvalues $\lambda^1,...,\lambda^{m-1}$ are equal, we call the state *uniaxial* and we can write

$$Q_p = \lambda^1 g^{\sharp\sharp} - m\lambda^1 E_m\otimes E_m = -m\lambda^1\left(E_m\otimes E_m - \frac{1}{m}g^{\sharp\sharp}\right)\;.$$

We call $s := -m\lambda^1$ the *order parameter* and $n_p = E_m$ the director, so that

$$Q_p = s\left(n_p\otimes n_p - \frac{1}{m}g^{\sharp\sharp}\right)\;. \tag{4}$$

In order to understand this term *order parameter*, consider the expected value of the following term. Here, θ is the angle between the direction of a molecule v and the director n:

$$\begin{aligned}\left\langle (\cos\theta)^2 - \frac{1}{m}\,\mathrm{d}\mu(v)\right\rangle &= \int_{\mathbb{S}_p M} \langle v, n\rangle_g^2 - \frac{1}{m}\,\mathrm{d}\mu(v)\\ &= \int_{\mathbb{S}_p M} \left\langle v\otimes v, -\frac{1}{m} g^{\sharp\sharp}, n\otimes n\right\rangle_g \mathrm{d}\mu(v)\\ &\overset{(3)}{=} \langle Q, n\otimes n\rangle_g = \frac{m-1}{m} s\ .\\ \Longrightarrow\ s = \frac{m}{m-1}\left\langle (\cos\theta)^2 - \frac{1}{m}\right\rangle &\in \left[-\frac{1}{m-1}, 1\right]\ .\end{aligned}$$

The lower bound $s = -\frac{1}{m-1}$ means that all molecules are perpendicular to n, $s = 0$ means that the liquid is isotropic and $s = 1$ means that all molecules are perfectly aligned with n. According to [MN14, p. 3], a typical liquid crystal has $s = 0.6$ (here $m = 3$.)

Remark 2.2 (Constant order parameter). Throughout the thesis we assume the order parameter $s \in [-\frac{1}{m-1}, 1] \setminus \{0\}$ to be a constant. Of course, this is a significant restriction but still covers a range of use cases with homogeneous conditions like constant temperature, pressure and water content. Furthermore, studying the models with constant order parameter can prepare research for more complex models.

While we only consider the case of uniaxial nematic crystals with constant order parameter s we see that the Q-tensor model in the general form fixes two other shortcomings of the Frank–Oseen model. For once, the director field does take into account how much the molecules vary around the direction given by the director. This is also related to the fact that liquid crystals are in some cases forced to have ‘defects’ where no director can be properly defined. That can be modeled with $s = 0$. Secondly, the Frank–Oseen model assumes that nematic liquid crystals are uniaxial but biaxial examples were also found. Note that we are considering any number of dimensions $m \in \mathbb{N}$ but since we live in three dimensions, the maximum of independent eigenvalues of Q can be 2, so there cannot be triaxial nematic liquid crystals.

Definition 2.3 (Constrained Q-tensors)**.** We introduce the following fiber bundles over M. Let $p \in M$. The unit vector bundle is denoted by

$$\mathbb{S}M := \left\{ n \in TM \,\middle|\, |n|_g = 1 \right\} \subset TM$$

with the fiber

$$\mathbb{S}_p M := \left\{ n \in T_p M \,\middle|\, |n|_g = 1 \right\} \subset T_p M \text{ as before.}$$

The constrained Q-tensor bundle is denoted by

$$\mathcal{Q}^{\mathbb{S}} M := \left\{ s\left(n\otimes n - \frac{1}{m} g^{\sharp\sharp}\right) \,\middle|\, n \in \mathbb{S}M \right\} \subset \mathcal{Q}M$$

with the fiber

$$\mathcal{Q}^{\mathbb{S}}_p M := \left\{ s\left(n\otimes n - \frac{1}{m} g^{\sharp\sharp}\right) \,\middle|\, n \in \mathbb{S}_p M \right\} \subset \mathcal{Q}_p M\ . \qquad \blacktriangleleft$$

Connected to those definitions is the projection operator.

Definition 2.4 (Projection operator)**.** The mapping from a direction to a constrained Q-tensor is called the *projection operator*.

$$P\colon \mathbb{S}M \to \mathcal{Q}^{\mathbb{S}}M$$
$$P(n) := s\left(n \otimes n - \frac{1}{m} g^{\sharp\sharp}\right)$$

By abuse of notation, we also call $P \circ \cdot$ just P:

$$P\colon \Gamma_R(\mathbb{S}M) \to \Gamma_{R'}(\mathcal{Q}^{\mathbb{S}}M)$$
$$(P(n))_p := s\left(n_p \otimes n_p - \frac{1}{m} g_p^{\sharp\sharp}\right) \quad (p \in M)$$

Here, R and R' are a regularity classes like C for continuous, C^∞ for smooth, $W^{\nabla,q}$ for Sobolev or $W^{\frac{1}{2},2}$ for fractional Sobolev. ◀

We will see that in our cases the line field $P(n)$ will be as regular as the unit vector field n: $R = R'$.

Remark 2.5 (Line fields)*.* We see that $P(-n) = P(n)$ since $(-n) \otimes (-n) = (-1)^2 n \otimes n$. As the introduction explains, the identification of two opposite unit vectors defines a line. Therefore, we will call the constrained Q-tensor fields $\Gamma_R(\mathcal{Q}^{\mathbb{S}}M)$ *line fields*.

Definition 2.6 (Orientable line field)**.** A line field $Q \in \Gamma_R(\mathcal{Q}^{\mathbb{S}}M)$ is called *orientable* if there exists $n \in \Gamma_R(\mathbb{S}M)$ such that $P(n) = Q$. In this case n is called an *orientation of* Q. Here R is a regularity class as in Definition 2.4. ◀

3 Sobolev spaces

In the introduction, we have discussed that the minimization problems are solved in Sobolev spaces instead of spaces of continuous fields. This section presents how Sobolev spaces of vector and line fields can be defined. The first Section 3.1 defines them and the norms on the Sobolev spaces intrinsically. The other two subsections use an embedding and a global frame to define the Sobolev spaces and show that these definitions are equivalent to the intrinsic one. We use those non-intrinsic views on Sobolev spaces in the proofs in Section 5.

3.1 Sobolev tensor fields

This section defines weakly differentiable tensor fields on manifolds intrinsically. It is based on the definitions of Güneysu [Chapter I and III Gün17] but specialises to our cases of the covariant derivative ∇ on TM, and $T^{(2,0)}M$.

In the Euclidean case Sobolev spaces are defined such that all weak derivatives are in L^q. By summing up the L^q-norms of all derivatives we get the norm on those Sobolev spaces. It is possible to list 'all weak derivatives' by listing the partial derivatives in the coordinate directions. In the manifold setting this norm would depend on the local choice of coordinates to define the partial derivatives. Hence it would not be intrinsic. Therefore we define Sobolev spaces with respect to a partial differential operator. In the general case of [Gün17] several operators of arbitrary order can be combined to define a

Sobolev space. Here we only need ∇. Note that ∇ denotes different operators as seen in Example 3.2, depending on the rank of the tensors it operates on.

Throughout this section E and F are smooth metric vector bundles over the Riemannian compact manifold M. In our cases of interest E will be TM or $T^{(2,0)}TM$ and $F = E \otimes T^*M$.

Definition 3.1 (First-order partial differential operator)**.** An $\mathbb{R}$-linear map

$$D \colon \Gamma_{C^\infty}(E) \to \Gamma_{C^\infty}(F)$$

is called a *first-order smooth partial differential operator* if for any chart $((x_1, \dots, x_m), U)$ of M which admits frames $(e_1, \dots, e_s \in \Gamma_{C^\infty}(E|_U),\ f_1, \dots, f_t \in \Gamma_{C^\infty}(F|_U)$, there are (necessarily uniquely determined) smooth functions

$$D^\alpha : U \to \mathbb{R}^{s \times t} \text{ (matrices)}$$

where $\alpha \in \{0, 1, ..., m\}$such that for all $(\varphi^{(1)}, \dots, \varphi^{(s)}) \in C^\infty(U, \mathbb{R}^s)$ we have

$$D|_U \left(\varphi^{(i)} e_i\right) = (D^0)_i^j \varphi^{(i)} f_j + (D^\alpha)_i^j \frac{\partial \varphi^{(i)}}{\partial x^\alpha} f_j \text{ in } U \,.$$

We call the set of smooth first-order partial differential operators $\mathcal{D}(E, F)$. ◀

One well-known example of a partial differential operator is the exterior derivative, as discussed in Example I.5 of [Gün17]. In this thesis we consider connections.

Example 3.2 (Connection). Consider a connection ∇ with connection coefficients Γ_{ik}^k with respect to a local frame $(\frac{\partial}{\partial x^i})$ with dual coframe $(\mathrm{d}x^i)$. On a chart domain $U \subseteq M$ with coordinates x the definition of ∇ then reads

$$\begin{aligned} &E = TM, F = T^{(1,1)}TM, X \in \Gamma_{C^\infty}(TM) : \\ &\qquad \nabla|_U X^i \frac{\partial}{\partial x^i} = \left(\Gamma_{ki}^j X^i + \frac{\partial X^j}{\partial x^k}\right) \frac{\partial}{\partial x^j} \otimes \mathrm{d}x^k \,. \end{aligned}$$

With the notation of Definition 3.1 this is

$$\begin{aligned} e_i &= \frac{\partial}{\partial x^i}, & f_{jk} &= \frac{\partial}{\partial x^j} \otimes \mathrm{d}x^k, \\ (D^0)_i^{jk} &= \Gamma_{ki}^j, & (D^\alpha)_i^{jk} &= \delta^{\alpha k} \delta_i^j \,. \end{aligned}$$

For $(2, 0)$ tensors ∇ reads in local coordinates

$$\begin{aligned} &E = T^{(2,0)}TM, F = T^{(2,1)}TM, Y \in \Gamma_{C^\infty}(E) : \\ &\qquad \nabla|_U Y^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} = \left(\Gamma_{mi}^k Y^{il} + \Gamma_{mj}^l Y^{kj} + \frac{\partial Y^{kl}}{\partial x^m}\right) \frac{\partial}{\partial x^k} \otimes \frac{\partial}{\partial x^l} \otimes \mathrm{d}x^m \,. \end{aligned}$$

With the notation of Definition 3.1 this is

$$\begin{aligned} e_{ij} &= \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}, & f_{klm} &= \frac{\partial}{\partial x^k} \otimes \frac{\partial}{\partial x^l} \otimes \mathrm{d}x^m, \\ (D^0)_{ij}^{klm} &= \Gamma_{mi}^k \delta_j^l + \Gamma_{mj}^l \delta_i^k, & (D^\alpha)_{ij}^{klm} &= \delta_m^\alpha \delta_i^k \delta_j^l \end{aligned}$$

To define weak derivatives analogously to the Euclidean case we need the definition of the adjoint differential operator.

Definition 3.3 (Formal adjoint)**.** For any $D \in \mathcal{D}(E,F)$, there exists a unique $D^* \in \mathcal{D}(F,E)$ that satisfies

$$\int_M \langle D^*\psi, \phi\rangle_g \, \mathrm{d}V_g = \int_M \langle \psi, D\phi\rangle_g \, \mathrm{d}V_g$$

for all $\psi \in \Gamma_{C^\infty}(F)$ and $\phi \in \Gamma_{C^\infty}(E)$ with at least one of ψ and ϕ compactly supported. Here g denotes the metrics both on E and F. The operator D^* is called the *formal adjoint of D with respect to the metric g.* ◀

Güneysu [Proposition and definition I.7 Gün17] proves that this is well-defined and gives a formula in coordinates.

Example 3.4 (Adjoint of the connection)*.* To find the coefficients of ∇^* on vector fields in local coordinates we deduce it via integration by parts. For this let $\psi = \psi^i_j \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j$ and $\phi = \phi^a \frac{\partial}{\partial x^a}$ with at least one of ψ, ϕ with compact support within the chart domain. Then

$$\begin{aligned}
\int_M \langle \psi, \nabla\phi\rangle_g \, \mathrm{d}V_g &= \int_M \left\langle \psi^i_j \frac{\partial}{\partial x^i} \otimes \mathrm{d}x^j, \left(\Gamma^b_{ca}\phi^a + \frac{\partial \phi^b}{\partial x^c}\right) \frac{\partial}{\partial x^b} \otimes \mathrm{d}x^c \right\rangle_g \mathrm{d}V_g \\
&= \int_M g_{ib} g^{jc} \psi^i_j \left(\Gamma^b_{ca}\phi^a + \frac{\partial \phi^b}{\partial x^c}\right) \mathrm{d}V_g \\
&= \int_M g_{ib} g^{jc} \psi^i_j \Gamma^b_{ca}\phi^a - g_{ib} g^{jc} \frac{\partial \psi^i_j}{\partial x^c} \phi^b \, \mathrm{d}V_g && \text{(Integration by parts)} \\
&= \int_M \left(g_{ib} g^{jc} \psi^i_j \Gamma^b_{ca} - g_{ia} g^{jc} \frac{\partial \psi^i_j}{\partial x^c} \right) \phi^a \, \mathrm{d}V_g && \text{(rename } b \rightsquigarrow a\text{)} \\
&= \int_M \left(g_{ib} g^{jc} \psi^i_j \Gamma^b_{cd} g^{dk} g_{ka} - g_{ka} g^{jc} \frac{\partial \psi^k_j}{\partial x^c} \right) \phi^a \, \mathrm{d}V_g && (g^{dk} g_{ka} = \delta^d_a, i \rightsquigarrow k) \\
&= \int_M \left\langle \left(g_{ib} g^{jc} \psi^i_j \Gamma^b_{cd} g^{dk} - g^{jc} \frac{\partial \psi^k_j}{\partial x^c} \right) \frac{\partial}{\partial x^k}, \phi^a \frac{\partial}{\partial x^a} \right\rangle_g \mathrm{d}V_g \, .
\end{aligned}$$

$$\text{Hence } \nabla^*\psi = g^{jc} \left(g_{ib} g^{dk} \Gamma^b_{cd} \psi^i_j - \frac{\partial \psi^k_j}{\partial x^c} \right) \frac{\partial}{\partial x^k} \, .$$

∇^* for tensor fields can be derived similarly.

We use the adjoint to define a weak derivative with respect to a partial differential operator.

Definition 3.5 (Weak derivative)**.** Let $D \in \mathcal{D}(E,F)$ and $f \in \Gamma_{L^1_{\mathrm{loc}}}(E)$. We say that Df exists weakly if there is some $h \in \Gamma_{L^1_{\mathrm{loc}}}(F)$ such that

$$\int_M \langle D^*\psi, f\rangle_g \, \mathrm{d}V_g = \int_M \langle \psi, h\rangle_g \, \mathrm{d}V_g \quad \text{for all} \quad \psi \in \Gamma_{C^\infty_c}(F) \, .$$

We write $Df := h$. ◀

The weak derivative h is uniquely determined as shown in [Proposition and definition I.7 Gün17].

Now we can define Sobolev spaces of vector and tensor fields on manifolds.

Definition 3.6 (Sobolev space of vector bundle sections)**.** Let $q \in [1, \infty]$ and let E and F be smooth metric vector bundles. Let $D \in \mathcal{D}(E, F)$. Then, the Banach space

$$\Gamma_{W^{D,q}}(E) := \{f \in \Gamma_{L^q}(E) \,|\, Df \text{ exists and } Df \in \Gamma_{L^q}(F)\}$$

with the norm

$$\|f\|_{W^{D,q}} := \left(\|f\|_{L^q}^q + \|Df\|_{L^q}^q\right)^{\frac{1}{q}} = \left(\int_M |f|_g^q + |Df|_g^q \,\mathrm{d}V_g\right)^{\frac{1}{q}}$$

is called the *D-Sobolev space of L^q-sections* of E. ◀

The following results about basic properties of $\Gamma_{W^{D,q}}(E)$ are proven by Güneysu [Theorem I.19 and paragraph in front Gün17]:

Theorem 3.7 (Separability and reflexivity)**.** *For any $D \in \mathcal{D}(E, F)$, the spaces $\Gamma_{W^{D,q}}(E)$ are separable for all $q \in [1, \infty)$ and reflexive for all $q \in (1, \infty)$.*

It is also possible to define Sobolev spaces as the completion of smooth functions with respect to Sobolev norms. The Meyers–Serrin Theorem states that this is an equivalent definition. Güneysu [Gün17] also showed that this result also holds for vector bundle sections on manifolds.

Theorem 3.8 (Meyers–Serrin theorem (Density of smooth sections))**.** *In the situation of Definition 3.6, let $q \in [1, \infty)$. Then, for any $f \in \Gamma_{W^{D,q}}(E)$, there exists a sequence $(f_n)_n$ in $\Gamma_{C^\infty}(E) \cap \Gamma_{W^{D,q}}(E)$ which can be chosen in $\Gamma_{C_c^\infty}(E)$ if* supp f *is compact, such that*

$$|f_n(x)| \le \|f\|_{L^\infty} \in [0, \infty] \text{ for all } x \in M, n \in \mathbb{N}$$
$$\|f_n - f\|_{W^{D,q}} \to 0 \text{ as } n \to \infty \;.$$

We say that the smooth sections are dense in $\Gamma_{W^{D,q}}(E)$ and write

$$\overline{\Gamma_{C^\infty}(E) \cap \Gamma_{W^{D,q}}(E)}^{W^{D,q}} = \Gamma_{W^{D,q}}(E) \;.$$

3.2 Sobolev spaces defined by embedding the base manifold

The definitions of Sobolev tensor fields and the norm in Section 3.1 have the advantage of being intrinsic. On the other hand, they have the disadvantage that in every point the tensor field, viewed as a function from the base manifold M, maps into a different space. This means that we cannot use the theory of mappings between manifolds. In order to use Theorem 5.1 about the approximation of mappings between manifolds we define Sobolev tensor field spaces differently and show that both definitions are equivalent.

By the Nash Embedding Theorem [Nas56] we can embed the base manifold M isometrically into $\mathbb{R}^N$ for some $N \in \mathbb{N}$. This embedding is called ι throughout this section. Then $\iota(M)$ is an embedded submanifold of $\mathbb{R}^N$ and thus, each T_pM ($p \in M$) is embedded into $\mathbb{R}^N$ via $\mathrm{d}\iota$.

As in [Example 4.8 Lee18, p. 92], we denote the Euclidean connection in $\mathbb{R}^N$ by $\overline{\nabla}$. That is for $X \in \Gamma_{C^\infty}(TM)$, a chart $\varphi\colon M \supseteq U \to \varphi(U) \subset \mathbb{R}^m$ and an index $i \in \{1, ..., m\}$

$$\overline{\nabla}_{\frac{\partial}{\partial \varphi^i}}(\mathrm{d}\iota \circ X) = \begin{pmatrix} \frac{\partial}{\partial \varphi^i}(\mathrm{d}\iota \circ X)^1 \\ \vdots \\ \frac{\partial}{\partial \varphi^i}(\mathrm{d}\iota \circ X)^N \end{pmatrix} \;.$$

In order to distinguish the scalar product in $\mathbb{R}^N$ and the metric g we denote the standard scalar product with a dot $\cdot$ such that $g(v,w) = \langle v,w \rangle_g = \mathrm{d}\iota(v) \cdot \mathrm{d}\iota(w)$ for all $v,w \in T_pM$ and $p \in M$. In particular $\left|\mathrm{d}n_p\right|_{\mathbb{R}^N} = 1$ if $\left|n_p\right|_g = 1$ (for any $p \in M$, $n_p \in \mathbb{S}_pM$). That means that we can consider all unit vectors at different points $p \in M$ as elements of the common space $\mathbb{S}^{N-1} = \left\{ v \in \mathbb{R}^N \,\middle|\, |v|_{\mathbb{R}^N} = 1 \right\}$ since $\mathrm{d}\iota(\mathbb{S}_pM) \subseteq \mathbb{S}^{N-1}$.

We would like to have the same for lines. In Section 2.3 we introduced the constrained Q-tensor bundle $\mathcal{Q}^{\mathbb{S}}M$. We saw that they model lines (Note 2.5). If we apply the embedding to the constrained Q-tensors we see that they are of the form $s(d\iota_p n_p \otimes d\iota_p n_p - \frac{1}{m} d\iota_p g_p^{\sharp\sharp}) \in T^{2,0}\mathbb{R}^N$ with some $n_p \in \mathbb{S}_pM$ at some point $p \in M$. Therefore we would like to consider all tangent line spaces as subsets of $\mathcal{Q}^{\mathbb{S}}\mathbb{R}^N$ like

$$\mathrm{d}\iota(\mathcal{Q}^{\mathbb{S}}_pM) \subseteq \mathcal{Q}^{\mathbb{S}}\mathbb{R}^N \overset{?}{=} \left\{ s\left(n \otimes n - \frac{1}{m}\,\mathrm{d}\iota \circ g^{\sharp\sharp}\right) \,\middle|\, n \in \mathbb{S}^{N-1} \right\}$$

but this is an invalid definition because g depends on the point on M whereas $\mathcal{Q}^{\mathbb{S}}\mathbb{R}^N$ should be a common line space for all points $p \in M$. We solve this by removing the $g^{\sharp\sharp}$ in the definition of lines as presented in the following definition. This results in a very similar model for lines.

Definition 3.9 (Sobolev spaces based on embedding the base manifold)**.** The Sobolev spaces based on embedding M into a Euclidean space are for $q \in [1,\infty)$ defined as

$$\Gamma_{W^{1,q}}(TM) := \left\{ v \in W^{1,q}(M,\mathbb{R}^n) \,\middle|\, v_p \in \mathrm{d}\iota(T_pM) \text{ for almost all } p \in M \right\} \tag{5}$$

$$W^{1,q}(M,\mathbb{S}^{N-1}) := \left\{ n \in W^{1,q}(M,\mathbb{R}^n) \,\middle|\, |n|_{\mathbb{R}^N} = 1 \right\} \tag{6}$$

$$\Gamma_{W^{1,q}}(\mathbb{S}M) := W^{1,q}\left(M,\mathbb{S}^{N-1}\right) \cap \Gamma_{W^{1,q}}\left(TM\right) \tag{7}$$

$$\begin{aligned} W^{1,q}\left(M,\mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N\right) := \big\{ Q \in W^{1,q}\left(M,T^{(2,0)}\mathbb{R}^N\right) \,\big|\, Q_p = n \otimes n \text{ for some} \\ n \in \mathbb{S}^{N-1} \text{ for almost every } p \in M \big\} \end{aligned} \tag{8}$$

$$\begin{aligned} W^{1,q}\left(M,T^{(2,0)}M\right) := \big\{ Q \in W^{1,q}\left(M,T^{(2,0)}\mathbb{R}^N\right) \,\big|\, Q_p(\eta,\cdot) = Q_p(\cdot,\eta) = 0 \\ \text{for all } \eta \perp \mathrm{d}\iota(T_pM) \text{ for almost every } p \in M \big\} \end{aligned} \tag{9}$$

$$\Gamma_{W^{1,q}}\left(\mathcal{Q}^{\mathbb{S}'}M\right) := W^{1,q}(M,\mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N) \cap \Gamma_{W^{1,q}}\left(T^{(2,0)}M\right)\ . \tag{10}$$

We identify tangent line fields defined intrinsically with tangent line fields defined with the embedding ι via

$$\begin{aligned} \iota_Q \colon \Gamma_{W^{\nabla,q}}(\mathcal{Q}^{\mathbb{S}}M) &\to \Gamma_{W^{1,q}}(\mathcal{Q}^{\mathbb{S}'}M) \\ \iota_Q(Q) &:= \iota_*\left(\frac{Q}{s} + \frac{1}{m}g^{\sharp\sharp}\right)\ . \end{aligned} \tag{11}$$

We also extend the definition of the projection operator P to unit vector fields that are not necessarily tangent to M. Here we use as in Definition 2.4 the same notation P_N for two closely related maps. It will be clear from the context which one is used.

$$\begin{aligned} P_N \colon \mathbb{S}^{N-1} &\to \mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N \\ P_N(n) &:= n \otimes n \\ P_N \colon W^{1,q}\left(M,\mathbb{S}^{N-1}\right) &\to W^{1,q}\left(M,\mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N\right) \\ (P_N(n))(p) &:= n(p) \otimes n(p) \qquad \text{(for all } p \in M) \end{aligned} \tag{12}$$

◀

The definition of P_N claims, that it maps into $W^{1,q}(M, \mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N)$. This will be justified in Lemma 3.11 in basically the same way as in Lemma 1 of [BZ11].

In order to calculate with with classical derivatives we need the ACL characterisation. *ACL* stands for 'absolutely continuous on lines'.

Theorem 3.10 (Nikodym, ACL characterization)**.** *(from [Theorem 1.49 Kin21]) Assume that $u \in W^{1,q}_{loc}(\Omega)$, $1 \le q \le \infty$ and let $\Omega' \Subset \Omega$ be compactly embedded in Ω. Then, there exists $u^* \colon \Omega \to [-\infty, \infty]$ such that $u^* = u$ almost everywhere in Ω and u^* is absolutely continuous on $(n-1)$-dimensional Lebesgue measure almost every line segments in Ω' that are parallel to the coordinate axes and the classical partial derivatives of u^* coincide with the weak partial derivatives of u almost everywhere in Ω. Conversely, if $u \in L^q_{loc}(\Omega)$ and there exists u^* as above such that $Du^* \in L^q_{loc}(\Omega)$, $i = 1, ..., \dim \Omega$, then $u \in W^{1,q}_{loc}(\Omega)$.*

Lemma 3.11 (Image of embedding based projection)**.** *Let $q \in [1, \infty)$. For a Sobolev unit vector field $n \in W^{1,q}(M, \mathbb{S}^{N-1})$, the corresponding line field $Q = P_N(n)$ is in $W^{1,q}(M, \mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N)$. Conversely, if $P_N(n) = Q \in W^{1,q}(M, \mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N)$ and n is continuous, then $n \in W^{1,q}(M, \mathbb{S}^{N-1})$.*

Proof. For the first part let $n \in W^{1,q}(M, \mathbb{S}^{N-1})$. The component functions of $Q = P(n)$ are products of components of n and thus weakly differentiable. Furthermore in a chart domain $U \subseteq M$ with orthonormal frame $(E_i)_i$ and coframe $(\varepsilon^i)_i$ we have

$$\overline{\nabla} Q = \overline{\nabla}_{E_i} n \otimes n \otimes \varepsilon^i + n \otimes \overline{\nabla}_{E_i} n \otimes \varepsilon^i \ .$$

Since $|n|_g$ is bounded, $\overline{\nabla} Q \in L^q(M, T^{(2,0)}\mathbb{R}^N \otimes T^*M)$ and Q is bounded. Therefore, $Q \in W^{1,q}(M, \mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N)$.

For the second part assume $P_N(\hat{n}) = \hat{Q} \in W^{1,q}(M, \mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N)$ and $\hat{n} \in C(M, \mathbb{S}^{N-1})$. Take a chart $\varphi \colon M \supseteq U \to \mathbb{R}^m$ and consider $\hat{Q}$ and $\hat{n}$ in those coordinates: $Q := \hat{Q}\big|_U \circ \varphi^{-1}$ and $n := \hat{n}|_U \circ \varphi^{-1}$. For Q we use the ACL characterization Theorem 3.10 and write Q for Q^* without loss of generality. For almost any point $p \in \varphi(U)$ and a direction e_k we consider the difference ($t \in \mathbb{R}$ sufficiently small)

$$\begin{aligned} Q^{ij}(p + te_k) - Q^{ij}(p) &= n^i(p + te_k) n^j(p + te_k) - n^i(p) n^j(p) \\ &= n^i(p + te_k) \left(n^j(p + te_k) - n^j(p) \right) + \left(n^i(p + te_k) - n^i(p) \right) n^j(p) \end{aligned}$$

When multiplying with $\frac{1}{2}\left(n^j(p + te_k) + n^j(p) \right)$ and summing over j we use that the first term vanishes because of

$$\sum_{j=1}^{N} \left(n^j(p + te_k) - n^j(p) \right) \left(n^j(p + te_k) + n^j(p) \right) = \sum_{j=1}^{N} n^j(p + te_k)^2 - n^j(p)^2 = 1 - 1 = 0$$

and get after dividing by t

$$\begin{aligned} &\frac{1}{t} \sum_{j=1}^{N} \left(\left(Q^{ij}(p + te_k) - Q^{ij}(p) \right) \cdot \frac{1}{2} \left(n^j(p + te_k) + n^j(p) \right) \right) \\ =&\frac{1}{t} \sum_{j=1}^{N} \left(n^i(p + te_k) - n^i(p) \right) n^j(p) \cdot \frac{1}{2} \left(n^j(p + te_k) + n^j(p) \right) \ . \end{aligned}$$

When we let $t \to 0$ and use that Q is differentiable and n is continuous, we get

$$\lim_{t \to 0} \frac{n^i(p + te_k) - n^i(x)}{t} = \sum_{j=1}^{N} \partial_k Q^{ij}(p) n^j(p) \ .$$

This shows that the partial classical derivatives of n exist a. e. in $\varphi(U)$ and satisfy

$$\partial_k n^i = \sum_{j=1}^{N} \partial_k Q^{ij} n^j, \tag{13}$$

and since $\overline{\nabla} Q \in L^q$ it follows from the ACL characterization that $n \in W^{1,q}(\varphi(U), \mathbb{S}^{N-1})$ and hence since U was arbitrary, $\hat{n} \in W^{1,q}(M, \mathbb{S}^{N-1})$. □

In order to relate differentiability in $\mathbb{R}^N$ with intrinsic covariant differentiability on M, we need to know how the Levi-Civita connection looks on the embedded manifold.

Proposition 3.12 (Levi-Civita connection of embedding)**.** *The Levi-Civita connection on M is given by $\nabla = \iota^* \nabla^T$ with ∇^T being the tangential connection as defined in Lee [(4.4) Lee18, p. 87]:*

$$\nabla^T_v Y := \pi^T \left(\overline{\nabla}_v \tilde{Y} \right)$$

where π^T is the projection onto the tangent space, $\overline{\nabla}$ is the Euclidean connection in the surrounding space and $\tilde{Y}$ is an extension of the vector field Y to the surrounding space.

Proof. Lee [Example 4.9 Lee18, p. 93] shows that the tangential connection is well-defined. By Lee [Propositon 5.12 (b) Lee18, p. 124] the tangential connection is the unique Levi-Civita connection of the embedded manifold $\iota(M)$. Also by Lee [Proposition 5.13 Lee18, p. 125] and since $\iota\colon M \to \iota(M)$ is an isometry, $\iota^* \nabla^T = \nabla$. □

Since $\nabla = \iota^* \nabla^T$, the intrinsic norm can be calculated in the embedding as

$$\|\iota_* X\|_{W^{\nabla,q}} = \sqrt[q]{\int_M |\iota_* X|^q_{\mathbb{R}^N} + |\nabla^T(\iota_* X)|^q_{\mathbb{R}^N} \, \mathrm{d}V_g} \qquad (X \in \Gamma_{W^{\nabla,q}}(TM)) \, .$$

On the other hand the norm on $\Gamma_{W^{1,q}}(TM)$ is

$$\|X\|_{W^{1,q}} = \sqrt[q]{\int_M |X|^q_{\mathbb{R}^N} + \left|\overline{\nabla} X\right|^q_{\mathbb{R}^N} \, \mathrm{d}V_g} \qquad (X \in W^{1,q}(TM)) \, .$$

In order to show equality of the definitions we need to show that those norms are equivalent.

Lemma 3.13 (Standard and tangential norm)**.** *Let $\iota\colon M \hookrightarrow \mathbb{R}^N$ be an isometric embedding of a compact Riemannian manifold M. Then, there exists a constant C depending on M, N and ι such that for all $X \in \Gamma_{C^\infty}(TM)$*

$$\|X\|_{W^{\nabla,q}} \leq \|\iota_* X\|_{W^{1,q}} \leq C \|X\|_{W^{\nabla,q}} \tag{14}$$

It follows that

$$\begin{aligned} \iota_* &\colon \Gamma_{W^{\nabla,q}}(TM) \to \Gamma_{W^{1,q}}(TM) \\ \iota_Q &\colon \Gamma_{W^{\nabla,q}}(\mathcal{Q}^{\mathbb{S}} M) \to \Gamma_{W^{1,q}}(\mathcal{Q}^{\mathbb{S}'} M) \end{aligned}$$

are continuous and bijective with continuous inverse. In the case of the linear operator ι_* this means that the norms are comparable and the spaces $\Gamma_{W^{\nabla,q}}(TM)$ and $\Gamma_{W^{1,q}}(TM)$ are isomorphic.

Proof. Let $p \in U \subseteq M$ be a chart domain with chart θ so that the coordinates on M are called $\theta^1, ..., \theta^m$ and the coordinates on $\mathbb{R}^N$ are called $x^1, ..., x^N$. Denote the ith component of the embedding by $\iota^i \colon \mathbb{R}^m \supseteq \theta(U) \to \mathbb{R}$ defined by $x^i \circ \iota = \iota^i \circ \theta$. Let $v = v^j \frac{\partial}{\partial \theta^j}$ be a local tangent vector field around p and w a direction, i. e. tangent vector, at p. Then

$$\begin{aligned}
\iota_* v = \mathrm{d}\iota\left(\frac{\partial}{\partial\theta^j}\right) &= \frac{\partial \iota^i}{\partial \theta^j}\frac{\partial}{\partial x^i} \\
\mathrm{d}\iota\left(v^j\frac{\partial}{\partial\theta^j}\right) &= \underbrace{v^j\frac{\partial \iota^i}{\partial \theta^j}}_{:=\tilde{v}^i}\frac{\partial}{\partial x^i} \\
\overline{\nabla}_w(\iota_* v) &= w(\tilde{v}^i)\frac{\partial}{\partial x^i} \\
&= w\left(v^j\frac{\partial \iota^i}{\partial \theta^j}\right)\frac{\partial}{\partial x^i} \\
&= \left(\underbrace{w(v^j)\frac{\partial \iota^i}{\partial \theta^j}}_{\text{scalar mult.}} + v^j \underbrace{w\left(\frac{\partial \iota^i}{\partial \theta^j}\right)}_{\text{twice diff.}}\right)\frac{\partial}{\partial x^i}
\end{aligned}$$

The tangential connection ∇^T_w is the orthogonal projection of the standard connection $\overline{\nabla}_w$ onto the tangent space. Call the orthogonal complement $\nabla^\perp_w$, such that

$$\left|\overline{\nabla}_w \iota_* v\right|^2_{\mathbb{R}^N} = \left|\nabla^T_w \iota_* v\right|^2_{\mathbb{R}^N} + \left|\nabla^\perp_w \iota_* v\right|^2_{\mathbb{R}^N} \ . \tag{15}$$

Note that this is an abuse of notation because $\nabla^\perp$ is not a connection.

Therefore, in order to compare the two norms we need to look at $\nabla^\perp_w v$. To calculate $\nabla^\perp_w$, take a local orthonormal frame of the normal bundle around $\iota(p)$ and call it $(\eta_l)_l$. Call the corresponding set of covectors η^l such that

$$\eta^l(\mathrm{d}\iota(v)) = 0 \text{ for all } v \in T_pM, \text{ that is } v(\iota^i){\eta^l}_i = 0 \tag{16}$$

$$\begin{aligned}
\text{Then } \nabla^\perp_w(v) = \eta^l(\overline{\nabla}_w v)\eta_l &= {\eta^l}_i\left(w(v^j)\frac{\partial \iota^i}{\partial\theta^j} + v^j w\left(\frac{\partial \iota^i}{\partial \theta^j}\right)\right)\eta_l \\
&= \left(w(v^j)\underbrace{\frac{\partial}{\partial\theta^j}(\iota^i){\eta^l}_i}_{\overset{(16)}{=}0} + {\eta^l}_i v^j w\left(\frac{\partial \iota^i}{\partial\theta^j}\right)\right)\eta_l \\
&= v^j \underbrace{{\eta^l}_i w\left(\frac{\partial \iota^i}{\partial\theta^j}\right)\eta_l}_{\text{indep. of } v}
\end{aligned} \tag{17}$$

When we plug in an orthonormal basis $(w_i)_i$ for w at each point, we can bound the term in the last line that is independent of v globally by some constant $C \geq 1$ since M is compact and η_l and ι are smooth. Hence for $v \in \Gamma_{C^\infty}(TM)$

$$\begin{aligned}
\|v\|^q_{W^{\nabla,q}} &= \int_M |v|^q_g + |\nabla v|^q_g \,\mathrm{d}V_g \\
&= \int_M |\iota_* v|^q_{\mathbb{R}^N} + \sum_{i=1}^m \left|\nabla^T_{w_i}(\iota_* v)\right|^q_{\mathbb{R}^N} \mathrm{d}V_g \\
&\overset{(15)}{\leq} \int_M |\iota_* v|^q_{\mathbb{R}^N} + \sum_{i=1}^m \left|\overline{\nabla}_{w_i}(\iota_* v)\right|^q_{\mathbb{R}^N} \mathrm{d}V_g
\end{aligned}$$

$$
\begin{aligned}
&= \|\iota_* v\|^q_{W^{1,q}} \\
&= \int_M |\iota_* v|^q_{\mathbb{R}^N} + \sum_{i=1}^m \left(\left|\nabla^T_{w_i}(\iota_* v)\right|^2_{\mathbb{R}^N} + \left|\nabla^\perp_{w_i}(\iota_* v)\right|^2_{\mathbb{R}^N} \right)^{\frac{q}{2}} \mathrm{d}V_g \\
&\leq \int_M |\iota_* v|^q_{\mathbb{R}^N} + \sum_{i=1}^m \left(\left|\nabla^T_{w_i}(\iota_* v)\right|^2_{\mathbb{R}^N} + C^2 |\iota_* v|^2 \right)^{\frac{q}{2}} \mathrm{d}V_g \\
&\leq \int_M |\iota_* v|^q_{\mathbb{R}^N} + 2^{\frac{q}{2}} \left(\sum_{i=1}^m \left|\nabla^T_{w_i}(\iota_* v)\right|^q_{\mathbb{R}^N} + |\iota_* v|^q C^q \right) \mathrm{d}V_g \\
&\overset{C\geq 1}{\leq} 2^{\frac{q}{2}} C^q \|v\|^q_{W^{\nabla,q}}
\end{aligned}
$$

So the norms are equivalent. The topology of M is the subspace topology as a submanifold of $\mathbb{R}^N$. Therefore, the definition of 'measurable function' is the same. Since smooth functions are dense in $\Gamma_{W^{\nabla,q}}(TM)$ by Theorem 3.8 and $W^{1,q}(M,\mathbb{R}^N)$ is complete and $W^{1,q}(TM)$ is a closed subspace of $W^{1,q}(M,\mathbb{R}^N)$, ι_* is a linear continuous map. It remains to show that $\iota_*(\Gamma_{W^{\nabla,q}}(TM)) = W^{1,q}(TM)$ and not a smaller subspace.

Lemma 3.14 (Density of smooth sections in embedding)**.** *The smooth sections $\Gamma_{C^\infty}(TM)$ are dense in $\Gamma_{W^{1,q}}(TM)$.*

Proof. The smooth functions $C^\infty(M,\mathbb{R}^N)$ are dense in $W^{1,q}(M,\mathbb{R}^N)$. Therefore, we can find for any $X \in \Gamma_{W^{1,q}}(TM)$ a sequence of smooth functions $X^{(k)} \in C^\infty(M,\mathbb{R}^N)$ converging to X in the $W^{1,q}$ norm. We want to show that by projecting the $X^{(k)}$ onto TM we get an approximating sequence of tangent vector fields. The projection π^T mentioned in Proposition 3.12 can locally be written as $\pi^T(v) = v - \eta^l(v)\eta_l$ with the local orthonormal frame of the normal bundle introduced above before (16). This shows that $\left|\pi^T(v)\right|_{\mathbb{R}^N} \leq |v|_{\mathbb{R}^N}$. Furthermore π^T is the identity on values of X and linear and therefore,

$$
\overline{\nabla}(\pi^T \circ X^{(k)} - X) = \overline{\nabla}(\pi^T \circ (X^{(k)} - X)) = D\pi^T(\overline{\nabla}(X^{(k)} - X)
$$

and since M and ι are smooth and M is compact, $\left|D\pi^T\right|_{\mathbb{R}^N}$ can be bound by some global constant C. Hence $\left\|\pi^T \circ X^{(k)} - X\right\|_{W^{1,q}} \leq C\left\|X^{(k)} - X\right\|_{W^{1,q}} \to 0$ as $k \to \infty$. $\pi^T \circ X^{(k)}$ is smooth since π^T is smooth. Therefore, X can be approximated by smooth tangent vector fields. □

Lemma 3.14 shows that, indeed, ι_* maps bijectively the completion $\Gamma_{W^{\nabla,q}}(TM)$ of $\Gamma_{C^\infty}(TM)$ onto the completion $\Gamma_{W^{1,q}}(TM)$ of $\Gamma_{C^\infty}(TM)$.

The proof for $T^{(2,0)}$ tensors is essentially the same, just with more indices. The local orthonormal frame of the normal bundle is of the form

$$
(\eta_l \otimes \varepsilon_r, \varepsilon_r \otimes \eta_l, \eta_l \otimes \eta_{l'})_{l,l'=1,\dots,N-m,r=1,\dots,m}
$$

with $(\eta_l)_l$ being the orthonormal frame of the normal bundle as before and $(\varepsilon_r)_r$ a local orthonormal frame of the tangent bundle.

The identifications ι_Q and ι_* are continuous with continuous inverse since $Q \mapsto \frac{Q}{s} + \frac{1}{m} g^{\sharp\sharp}$ is an affine transformation. □

3.3 Sobolev field spaces on parallelizable manifolds

We need connections to define differentiation of tensor fields on manifolds because the tangent spaces at different points cannot otherwise be identified in a natural manner. If the tangent bundle is trivial though, we have another way of identifying the tangent spaces. Manifolds with trivial tangent bundle are called *parallelizable*. In this case we can define Sobolev spaces based on this trivialization and will need this definition for the proof of Theorem 5.12. In this section we will show that the Sobolev space based on the trivialization is isomorphic to the intrinsic definition based on the connection from Section 3.1.

Within this section assume that the manifold M is parallelizable and choose a global orthonormal smooth frame $(B_1, ..., B_m)$.

Definition 3.15 (Sobolev field spaces on parallelizable manifold)**.** Define for $q \in [1, \infty)$

$$\begin{aligned} W^{1,q}_{\shortparallel}(TM) &:= \left\{ X \in \Gamma_{\mathcal{L}}(TM) \,\middle|\, \langle B_i, X \rangle_g \in W^{1,q}(M, \mathbb{R}) \ (i = 1, ..., m) \right\} \\ \|X\|_{W^{1,q}_{\shortparallel}} &:= \sqrt[q]{\sum_{i=1}^{m} \left\| \langle B_i, X \rangle_g \right\|^q_{W^{1,q}}} \\ W^{1,q}_{\shortparallel}(\mathcal{Q}M) &:= \left\{ Q \in \Gamma_{\mathcal{L}}(\mathcal{Q}M) \,\middle|\, \left\langle B_i \otimes B_j, Q \right\rangle_g \in W^{1,q}(M, \mathbb{R}) \ (i, j = 1, ..., m) \right\} \\ \|Q\|_{W^{1,q}_{\shortparallel}} &:= \sqrt[q]{\sum_{i,j=1}^{m} \left\| \left\langle B_i \otimes B_j, Q \right\rangle_g \right\|^q_{W^{1,q}}} \end{aligned}$$

◀

Lemma 3.16 (Equivalence of standard and trivialization norm)**.** *The norms $\|\cdot\|_{\Gamma_{W^{\nabla,q}}}$ and $\|\cdot\|_{W^{1,q}_{\shortparallel}}$ are equivalent on $\Gamma_{C^1}(TM)$. That means that there exists $C \geq 1$ such that for all $X \in \Gamma_{C^1}(TM)$,*

$$\frac{1}{C} \|X\|_{\Gamma_{W^{\nabla,q}}} \leq \|X\|_{W^{1,q}_{\shortparallel}} \leq C \|X\|_{\Gamma_{W^{\nabla,q}}} \ . \tag{18}$$

This implies that

$$W^{1,q}_{\shortparallel}(TM) \text{ and } \Gamma_{W^{\nabla,q}}(TM) \text{ are isomorphic} \tag{19}$$

$$\text{and } W^{1,q}_{\shortparallel}(\mathcal{Q}M) \text{ and } \Gamma_{W^{\nabla,q}}(\mathcal{Q}M) \text{ are isomorphic.} \tag{20}$$

Proof. We estimate the $W^{1,q}_{\shortparallel}$-norm against the intrinsic norm. This is possible since the frame is smooth on a compact domain and therefore, $|B_i|_g = 1$ and $\|\nabla B_i\|_{L_q}$ is bounded for every $i = 1, ...m$.

$$\begin{aligned} \|X\|^q_{W^{1,q}_{\shortparallel}} &= \sum_{i=1}^{m} \left\| \langle B_i, X \rangle_g \right\|^q_{L^q} + \sum_{i=1}^{m} \left\| \nabla \langle B_i, X \rangle_g \right\|^q_{L^q} \\ &= \|X\|^q_{L^q} + \sum_{i=1}^{m} \left\| \sum_{j=1}^{m} \left(\left\langle \nabla_{B_j} B_i, X \right\rangle_g + \left\langle B_j, \nabla_{B_j} X \right\rangle_g \right) B_j^\flat \right\|^q_{L^q} \end{aligned}$$

$$\left| \sum_{j=1}^{m} \left(\left\langle \nabla_{B_j} B_i, X \right\rangle_g + \left\langle B_j, \nabla_{B_j} X \right\rangle_g \right) B_j^\flat \right|^q_g$$

$$
\begin{aligned}
&= \left(\sum_{j=1}^{m} \left(\left\langle \nabla_{B_j} B_i, X \right\rangle_g + \left\langle B_j, \nabla_{B_j} X \right\rangle_g \right)^2 \right)^{\frac{q}{2}} \\
&\leq \left(\sum_{j=1}^{m} 2 \left(\left| \nabla_{B_j} B_i \right|_g^2 \cdot |X|_g^2 + \left| B_j \right|_g^2 \cdot \left| \nabla_{B_j} X \right|_g^2 \right) \right)^{\frac{q}{2}} \\
&\leq 2^{\frac{q}{2}} \left(|\nabla B_i|_g^2 \cdot |X|_g^2 + 1 \cdot |\nabla X|_g^2 \right)^{\frac{q}{2}} \\
&= 2^{\frac{q}{2}} \cdot 2^{\frac{q}{2}} \left(|\nabla B_i|_g^q \cdot |X|_g^q + |\nabla X|_g^q \right) \\
\implies \|X\|_{W_{\shortparallel}^{1,q}}^q &\leq \|X\|_{L^q}^q + 2^q \left(\sum_{i=1}^{m} \|\nabla B_i\|_{L^q}^q \right) \|X\|_{L^q}^q + 2^q m \|\nabla X\|_{L^q}^q \\
&\leq C \|X\|_{W^{\nabla,q}}^q
\end{aligned}
$$

To get the estimate in the other direction we write ∇X in the coordinates $B_1, ..., B_m$ with Christoffel symbols. The coordinates are called $X^i := \langle B_i, X \rangle_g$. The Christoffel symbols can be calculated from the B_i's and are therefore bounded on the compact domain M by some constant C.

$$
\|X\|_{W^{\nabla,q}}^q = \|X\|_{L^q}^q + \left\| \left(B_j(X^i) + \Gamma_{jk}^i X^k \right) B_i \otimes B_j^\flat \right\|_{L^q}^q
$$

$$
\begin{aligned}
\left| \left(B_j(X^i) + \Gamma_{jk}^i X^k \right) B_i \otimes B_j^\flat \right|_g^q &\leq \left| \sum_{i,j=1}^{m} \left(B_j(X^i) + C \sum_{k=1}^{m} X^k \right)^2 \right|^{\frac{q}{2}} \\
&\leq (m+1)^{\frac{q}{2}} \left(\sum_{i,j=1}^{m} \left(\left| B_j(X^i) \right|^2 + C^2 \sum_{k=1}^{m} (X^k)^2 \right) \right)^{\frac{q}{2}} \\
&= (m+1)^{\frac{q}{2}} \left(m^2 C^2 |X|_g^2 + \left| \nabla (X^i)_i \right|_g^2 \right)^{\frac{q}{2}} \\
&= (m+1)^{\frac{q}{2}} 2^{\frac{q}{2}} \left(m^q C^q |X|_g^q + \left| \nabla (X^i)_i \right|_g^q \right) \\
\implies \|X\|_{W^{\nabla,q}}^q &\leq (2m+2)^{\frac{q}{2}} (m^q C^q + 1) \|X\|_{W_{\shortparallel}^{1,q}}^q
\end{aligned}
$$

We know that the smooth sections are dense in $\Gamma_{W^{\nabla,q}}(TM)$ and $W_{\shortparallel}^{1,q}(TM)$ by Theorem 3.8 and Theorem I.19 of [Gün17] applied on $\mathbb{R}^m$ as the vector bundle. Therefore, the norm estimates carry over to the Sobolev spaces, proving the result for vector fields. For $(2,0)$-tensors the same argument holds, just the calculations are longer and imply different constants. □

3.3.1 Fractional Sobolev space and trace

The goal of Section 5.2 is to reduce the question of orientability on a surface to the orientability on loops. For continuous functions restricting to a lower-dimensional manifold is trivial. In the case of Sobolev functions the restriction is not immediately well-defined since a lower-dimensional manifold is a Lebesgue null set. So any Sobolev function can be redefined on the submanifold without changing the function. This problem is solved by *traces* which give a meaning to the restriction of a Sobolev function to a submanifold. The idea is to define the trace for smooth functions by restriction and show that this defines a continuous operator and can thus be continuously extended to all Sobolev functions.

The more simple form shows that the trace of a $W^{1,q}$ function on a submanifold with one dimension less is an L^q function. Unfortunately this is not sufficient as we will see in Section 5.2. Indeed, the functions that are traces are more regular than L^q but less regular than $W^{1,q}$. These function spaces are called *fractional Sobolev spaces*. There are four ways to define them which are all equivalent in the Hilbert space case $q = 2$.

- The Fourier transform turns differentiation into multiplication operators. Multiplication operators multiply the value of a function by a power of the argument. This can also be defined for fractional exponents.
- The image of the trace operator.
- All L^2-functions that have a finite Gagliardo–Slobodeckij norm:

  $$\|f\|_{t,2} := \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^2}{|x-y|^{2t+\dim\Omega}} \,\mathrm{d}x\,\mathrm{d}y < \infty \ .$$

  Here $0 < t < 1$ is the differentiation degree and Ω is the domain.
- Interpolation spaces between Sobolev spaces as in [7.57 AF03].

Since we use the same ideas as Ball and Zarnescu [BZ11] we will use their definition which they take from [BN95].

Definition 3.17 (Fractional Sobolev space)**.** (From [Example 2 BN95]) Let N be a Riemannian manifold of dimension n, $0 < t < 1$ the differentiation degree and $q \in [1, \infty)$. Then the intrinsic definition of the fractional Sobolev space $W^{t,q}(N, \mathbb{R})$ is

$$W^{t,q}(N,\mathbb{R}) := \left\{ u \in L^q(N,\mathbb{R}) \,\middle|\, \int_N \int_N \frac{|u(x) - u(y)|^q}{\operatorname{dist}(x,y)^{tq+n}} < \infty \right\} \tag{21}$$

with the norm

$$\|u\|_{W^{t,q}} := \left(\|u\|_{L^q}^q + \int_N \int_N \frac{|u(x) - u(y)|^q}{\operatorname{dist}(x,y)^{tq+n}} \right)^{\frac{1}{q}} \ . \tag{22}$$

Here $\operatorname{dist}(x, y)$ is the distance on N that is given by the infimum of lengths of paths from x to y. ◀

Other papers like [BW93] first define $W^{t,q}(\mathbb{R}^n, \mathbb{R})$ and then use charts and a partition of unity to reduce the manifold case to the flat one. Call this fractional Sobolev space $W^{t,q}_{\eta,\varphi}N, \mathbb{R}$ when it is based on a partition of unity η subordinate to an atlas φ. This space has a different Gagliardo–Slobodeckij norm but it stays comparable:

Lemma 3.18 (Equivalence of Gagliardo–Slobodeckij norms)**.** *The compact manifold N of dimension n is equipped with a finite smooth partition $(\eta_k)_{1\le k\le K}$ subordinate to a finite atlas $((U_k, \varphi_k))_{1\le k\le K}$. Let the exponent be $q \in [1, \infty)$ and the differentiation degree be $0 < t < 1$ such that $q(1 - t) - n \ge 0$. Then the norm on $C^\infty(N, \mathbb{R})$ induced by the partition of unity and the atlas*

$$\|u\|_{\eta,\varphi,W^{t,q}} := \left(\sum_{k=1}^{K} \left\|(\eta_k u) \circ \varphi_k^{-1}\right\|_{W^{t,q}}^q \right)^{\frac{1}{q}}$$

$$\text{where } \|w\|_{W^{t,q}} := \left(\|w\|_{L^q}^q + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|w(x) - w(y)|^q}{|x-y|^{tq+n}} \right)^{\frac{1}{q}} \quad \text{on } C^\infty(\mathbb{R}^n, \mathbb{R})$$

is equivalent to the intrinsic norm $\|\cdot\|_{W^{t,q}}$ *defined in* (22)*. Note that* $\operatorname{supp}(\eta_k u) \subset U_k$ *and therefore* $(\eta_k u) \circ \varphi_k^{-1}$ *can trivially be extended from* $\varphi_k(U_k) \subseteq \mathbb{R}^n$ *to the entire space* $\mathbb{R}^n$ *by zero.*

Remark 3.19 (Exponent condition)*.* The condition $q(1-t) - n \geq 0$ is fulfilled in our case of 1-dimensional submanifolds of surfaces because $2(1-\frac{1}{2}) - 1 = 0$.

Proof. First we see that the L^q-norm parts are comparable.

$$\begin{aligned}\|u\|_{\eta,\varphi,L^q} &= \sum_{k=1}^{K} \int_{\mathbb{R}^N} ((\eta_k u) \circ \varphi_k^{-1})^q = \sum_{k=1}^{K} \int_{U_k} (\eta_k u)^q \, \mathrm{d}V_g \\ &= \sum_{k=1}^{K} \int_N \eta_k^q u^q \, \mathrm{d}V_g \qquad = \int_N \left(\sum_{k=1}^{K} \eta_k^q \right) u^q \, \mathrm{d}V_g \ .\end{aligned}$$

For the summands we know $n_k^q \leq \eta_k$ since $q \geq 1$ and $\eta_k \leq 1$. On the other hand, for the sum we have the mean inequality that states

$$\left(\frac{\sum_{k=1}^{K} \eta_k^q}{K} \right)^{\frac{1}{q}} \geq \frac{\sum_{k=1}^{K} \eta_k}{K} = \frac{1}{K}$$

$$\text{and hence,} \qquad \sum_{k=1}^{K} \eta_k^q \geq K^{1-q} > 0 \ .$$

$$\text{Consequently,} \qquad K^{1-q} \|u\|_{L^q} \leq \|u\|_{\eta,\varphi,L^q} \leq \|u\|_{L^q} \ .$$

As the next step, we compare $|\varphi_k(x) - \varphi_k(y)|$ with $\operatorname{dist}(x, y)$ for any $x, y \in U_k$ for some $k \in \{1, ..., K\}$. Both values are the infimum of path lengths from one point to the other but with different way of calculating the path length. Since the number of maps K is finite and all maps are smooth, the differentials $D\varphi_k$ and $D\varphi_k^{-1}$ are uniformly bounded by some $C > 0$, independent of k. That implies that the length of paths γ in U_k are comparable to the length of the paths $\varphi_k \circ \gamma$ in $\varphi_k(U_k) \subseteq \mathbb{R}^n$. Without loss of generality, we assume that all geodesics between points of one chart domain lie within this chart domain, for example by shrinking the chart domains to geodesic balls. We write $\gamma : x \rightsquigarrow y$ for paths from x to y and estimate

$$\begin{aligned}\operatorname{dist}(x, y) &= \inf_{\gamma : x \rightsquigarrow y} \operatorname{len}(\gamma) \qquad \leq \inf_{\gamma : x \rightsquigarrow y} C \operatorname{len}(\varphi_k \circ \gamma) \\ &= C \inf_{\gamma : \varphi_k(x) \rightsquigarrow \varphi_k(y)} \operatorname{len}(\gamma) = C |\varphi_k(x) - \varphi_k(y)| \\ |\varphi_k(x) - \varphi_k(y)| &= \inf_{\gamma : \varphi_k(x) \rightsquigarrow \varphi_k(y)} \operatorname{len}(\gamma) \quad \leq \inf_{\gamma : \varphi_k(x) \rightsquigarrow \varphi_k(y)} C \operatorname{len}(\varphi_k^{-1} \circ \gamma) \\ &= C \inf_{\gamma : x \rightsquigarrow y} \operatorname{len}(\gamma) \qquad = C \operatorname{dist}(x, y) \ .\end{aligned}$$

This shows that we can switch between $|\varphi_k(x) - \varphi_k(y)|$ and $\operatorname{dist}(x, y)$ as long as x and y are in the same chart domain.

Now we can compare the double integral terms for some $k \in \{1, ..., K\}$. To estimate the term $\eta_k(x) - \eta_k(y)$, use that the derivatives of the η_k are bounded and therefore there exists a Lipschitz constant L such that for all $k \in \{1, ..., K\}$, $|\eta_k(x) - \eta_k(y)| \le L \operatorname{dist}(x, y)$.

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\left|(\eta_k u)\circ\varphi_k^{-1}(x) - (\eta_k u)\circ\varphi_k^{-1}(y)\right|^q}{|x-y|^{tq+n}} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad\le C^{tq+n}\int_{U_k}\int_{U_k} \frac{|(\eta_k u)(x) - (\eta_k u)(y)|^q}{\operatorname{dist}(x,y)^{tq+n}} \,\mathrm{d}V_g(x)\,\mathrm{d}V_g(y) \\
&\quad\le C^{tq+n}\int_{U_k}\int_{U_k} \frac{|\eta_k(x)(u(x)-u(y)) + (\eta_k(x)-\eta_k(y))u(y)|^q}{\operatorname{dist}(x,y)^{tq+n}} \,\mathrm{d}V_g(x)\,\mathrm{d}V_g(y) \\
&\quad\le C^{tq+n}2^{q-1}\int_{U_k}\int_{U_k} \frac{|\eta_k(x)(u(x)-u(y))|^q}{\operatorname{dist}(x,y)^{tq+n}} \\
&\qquad + \frac{|(\eta_k(x)-\eta_k(y))u(y)|^q}{\operatorname{dist}(x,y)^{tq+n}} \,\mathrm{d}V_g(x)\,\mathrm{d}V_g(y) \\
&\quad\le C^{tq+n}2^{q-1}\Bigg(\int_N\int_N \frac{|(u(x)-u(y))|^q}{\operatorname{dist}(x,y)^{tq+n}} \,\mathrm{d}V_g(x)\,\mathrm{d}V_g(y) \\
&\qquad + \int_{U_k} |u(y)|^q \int_{U_k} \frac{(L\operatorname{dist}(x,y))^q}{\operatorname{dist}(x,y)^{tq+n}} \,\mathrm{d}V_g(x)\,\mathrm{d}V_g(y)\Bigg) && (\eta_k \le 1) \\
&\quad= C^{tq+n}2^{q-1}\Bigg(\int_N\int_N \frac{|(u(x)-u(y))|^q}{\operatorname{dist}(x,y)^{tq+n}} \,\mathrm{d}V_g(x)\,\mathrm{d}V_g(y) \\
&\qquad + L^q \int_{U_k} |u(y)|^q \int_{U_k} \operatorname{dist}(x,y)^{q(1-t)-n} \,\mathrm{d}V_g(x)\,\mathrm{d}V_g(y)\Bigg) \\
&\quad\le C^{tq+n}2^{q-1}\Bigg(\int_N\int_N \frac{|(u(x)-u(y))|^q}{\operatorname{dist}(x,y)^{tq+n}} \,\mathrm{d}V_g(x)\,\mathrm{d}V_g(y) \\
&\qquad + L^q \|u\|_{L^q}^q \int_N (\operatorname{diam} N)^{q(1-t)-n} \,\mathrm{d}V_g(x)\Bigg) && (q(1-t)-n \ge 0)
\end{aligned}$$

This shows that $\|u\|_{\eta,\varphi,W^{t,q}} \le C' \|u\|_{W^{t,q}}$ for some C' that does not depend on u.

Now we need an estimate for the opposite direction: estimate $\|u\|_{W^{t,q}}$ by $\|u\|_{\eta,\varphi,W^{t,q}}$. Observe that for every $x \in N$ there exist a function η_k in the partition of unity that is greater or equal to $\frac{1}{K}$ at x. Therefore we can partition N in the following way:

$$V_k := \left\{ x \in M \,\middle|\, \eta_k \ge \frac{1}{K} \right\} \setminus \bigcup_{i=1}^{k-1} V_i \subseteq U_k \;.$$

Then V_k is measurable, we have $K\eta_k|_{V_k} \ge 1$ for all $k \in \{1, ..., K\}$ and $\bigcup_{k=1}^K V_k = N$. We estimate

$$\begin{aligned}
&\int_N\int_N \frac{|u(x)-u(y)|^q}{\operatorname{dist}(x,y)^{tq+n}} \,\mathrm{d}V_g(y)\,\mathrm{d}V_g(x) \\
&\quad\le \sum_{k=1}^K \int_{V_k}\int_N K\eta_k(x)\frac{|u(x)-u(y)|^q}{\operatorname{dist}(x,y)^{tq+n}} \,\mathrm{d}V_g(y)\,\mathrm{d}V_g(x) \\
&\quad= K\sum_{k=1}^K \int_{V_k}\int_N \frac{|\eta_k(x)u(x) - \eta_k(y)u(y) + \eta_k(y)u(y) - \eta_k(x)u(y)|^q}{\operatorname{dist}(x,y)^{tq+n}} \,\mathrm{d}V_g(y)\,\mathrm{d}V_g(x) \\
&\quad\le K2^{q-1}\Bigg(\sum_{k=1}^K \int_{V_k}\int_N \frac{|(\eta_k u)(x) - (\eta_k u)(y)|^q}{\operatorname{dist}(x,y)^{tq+n}} \,\mathrm{d}V_g(y)\,\mathrm{d}V_g(x)
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{V_k} \int_N \frac{|(\eta_k(y) - \eta_k(x))u(y)|^q}{\operatorname{dist}(x,y)^{tq+n}} \, \mathrm{d}V_g(y) \, \mathrm{d}V_g(x) \Bigg) && \text{(mean inequality)} \\
&\le K 2^{q-1} \Bigg(\sum_{k=1}^{K} \int_{U_k} \int_{U_k} \frac{|(\eta_k u)(x) - (\eta_k u)(y)|^q}{\operatorname{dist}(x,y)^{tq+n}} \, \mathrm{d}V_g(y) \, \mathrm{d}V_g(x) \\
&+ \int_N u(y)^q \int_{V_k} \frac{(L \operatorname{dist}(x,y))^q}{\operatorname{dist}(x,y)^{tq+n}} \, \mathrm{d}V_g(x) \, \mathrm{d}V_g(y) \Bigg) \\
&\le K 2^{q-1} \Bigg(\sum_{k=1}^{K} \int_{U_k} \int_{U_k} \frac{|(\eta_k u)(x) - (\eta_k u)(y)|^q}{\operatorname{dist}(x,y)^{tq+n}} \, \mathrm{d}V_g(y) \, \mathrm{d}V_g(x) \\
&+ L^q \|u\|_{L^q}^q \int_N (\operatorname{diam} N)^{q(1-t)-n} \, \mathrm{d}V_g(x) \, \mathrm{d}V_g(y) \Bigg) && (V_k \subseteq N)
\end{aligned}
$$

In the last two steps we used the same estimate as in the previous calculation based on the assumption $q(1-t) - n \ge 0$.

Together with the estimate for $\|u\|_{L^q}$, we have shown that $\|u\|_{W^{t,q}} \le C'' \|u\|_{\eta,\varphi,W^{t,q}}$ for some C'' that does not depend on u. This concludes the proof. □

Since smooth functions are dense in the fractional Sobolev spaces with any of the mentioned definitions, equivalent norms imply that the corresponding spaces are isomorphic. Additionally to Lemma 3.18, [BW93] shows that the space $W^{t,q}_{\eta,\varphi}(N,\mathbb{R})$ does not depend on the choice of the atlas and the partition of unity. So we know

$$W^{t,q}(N,\mathbb{R}) = W^{t,q}_{\eta,\varphi}(N,\mathbb{R}) = W^{t,q}_{\eta',\varphi'}(N,\mathbb{R})$$

for any two finite atlases φ, φ' on N with subordinate finite partitions of unity η, η'.

The definition of $W^{t,q}$ in [BW93] also differs from Definition 3.17 by using the Fourier transform to define fractional Sobolev spaces on $\mathbb{R}^n$ instead of the Gagliardo–Slobodeckij norm. Therefore, we need to take [NPV12] into account that shows that the definition with the Fourier transform is equivalent to the one with the Gagliardo–Slobodeckij norm. Note that the Fourier transform based definition is valid for any $t \ge 0$ whereas in the definition via the norm as above $0 < t < 1$. For $t \in \mathbb{Z}$ we use the usual Sobolev space definition instead of Definition 3.17.

We conclude that we can apply the Trace Theorem 11.4 from [BW93]:

Theorem 3.20 (Trace theorem)**.** *Let M be a compact Riemannian manifold as usual and N a $(m-1)$-dimensional submanifold. Let $t \ge \frac{1}{2}$. Then the mapping $u \mapsto u|_N$ from $C^\infty(M,\mathbb{R})$ to $C^\infty(N,\mathbb{R})$ can be extended to the whole space $W^{t,2}(M,\mathbb{R})$ by continuity. It provides a linear map $\operatorname{Tr}_N \colon W^{t,2}(M,\mathbb{R}) \to W^{t-\frac{1}{2},2}(N,\mathbb{R})$.*

We use this theorem with $t = 1$ and $\operatorname{Tr}_N \colon W^{1,2}(M,\mathbb{R}) \to W^{\frac{1}{2},2}(N,\mathbb{R})$.

For $\mathbb{R}^m$-valued functions we consider every component separately. Hence, for an

$(m-1)$-dimensional submanifold N of M, there exist continuous linear operators

$$\mathrm{Tr}_N \colon W^{1,2}_{\shortparallel}(TM) \to W^{\frac{1}{2},2}_{\shortparallel}(TM|_N) \tag{23}$$

$$\text{with } W^{\frac{1}{2},2}_{\shortparallel}(TM|_N) := \left\{ X \in \Gamma_{\mathcal{L}}(TM|_N) \,\middle|\, \langle B_i, X \rangle_g \in W^{\frac{1}{2},2}(N,\mathbb{R}) \, (i = 1, \ldots, m) \right\} \tag{24}$$

$$\mathrm{Tr}_N \colon W^{1,2}_{\shortparallel}(\mathcal{Q}M) \to W^{\frac{1}{2},2}_{\shortparallel}(\mathcal{Q}M|_N) \tag{25}$$

$$\text{with } W^{\frac{1}{2},2}_{\shortparallel}(\mathcal{Q}M|_N) := \left\{ Q \in \Gamma_{\mathcal{L}}(\mathcal{Q}M|_N) \,\middle|\, \langle B_i \otimes B_j, Q \rangle_g \in W^{\frac{1}{2},2}(N,\mathbb{R}) \, (i, j = 1, \ldots, m) \right\} \tag{26}$$

agreeing with the usual restriction on continuous tensor fields.

4 Orientability of continuous line fields

The projection operator P maps two opposite points on the sphere onto the corresponding line. Compare this with the construction of the real projective space $\mathbb{RP}^d$ ($d \in \mathbb{N}$): We start with the sphere $\mathbb{S}^d$ and identify opposite points. This construction also shows that the sphere is a two-sheet covering space of $\mathbb{RP}^d$. Similarly, $\mathcal{Q}^{\mathbb{S}}M$ is a fiber bundle with fiber $\mathbb{RP}^{m-1}$ and the projection $P \colon \mathbb{S}M \to \mathcal{Q}^{\mathbb{S}}M$ is a covering map.

Definition 4.1 (Covering map)**.** (from Hatcher [1.3 Covering Spaces Hat01, p. 56]) A *covering space* of a topological space X is a topological space $\tilde{X}$ together with a map $p \colon \tilde{X} \to X$ satisfying the following condition: Each point $x \in X$ has an open neighborhood U in X such that $p^{-1}(U)$ is a union of disjoint open sets in $\tilde{X}$, each of which is mapped homeomorphically onto U by p. The disjoint open sets in $\tilde{X}$ that project homeomorphically to U by p are called *sheets* of $\tilde{X}$ over U. ◀

Algebraic geometry can characterise orientable continuous line fields. In the language of algebraic topology an orientation of a line field is a *lift* from a map into $\mathcal{Q}^{\mathbb{S}}M$ to a map into the covering space $\mathbb{S}M$. In this chapter we specialise the theory of lifts to our case of line fields. This generalises a result by Ball and Zarnescu [BZ11, Theorem 1].

Since we want to lift from $\mathcal{Q}^{\mathbb{S}}M$ to $\mathbb{S}M$, we need to check if $P \colon \mathbb{S}M \to \mathcal{Q}^{\mathbb{S}}M$ is a covering map.

Lemma 4.2 (Projection as a covering map)**.** *The projection operators $P \colon \mathbb{S}M \to \mathcal{Q}^{\mathbb{S}}M$ and $P_N \colon \mathbb{S}^{N-1} \to \mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N$ ($N \in \mathbb{N}$) are covering maps with two sheets.*

Proof. P is continuous since $n \mapsto s\left(n \otimes n - \frac{1}{m} g^{\sharp\sharp}\right)$ ($m = \dim M$) is continuous as a map from TM to $T^{(2,0)}TM$. P_N is continuous since $n \mapsto n \otimes n$ is continuous as a map from $\mathbb{S}^{N-1}$ to $T^{(2,0)}\mathbb{R}^N$. For $n \in \mathbb{S}M$ or $\mathbb{S}^{N-1}$ choose a neighborhood U that has a diameter of less than 2 (measured in TM or $\mathbb{R}^N$ respectively). Then, $P^{-1}(P(U)) = U \cup -U$ (same for P_N) and U and $-U$ are disjoint because the distance between opposite points is 2. Since P and P_N are quadratic, they have continuous inverses if they are bijective. Hence they map U and $-U$ homeomorphically onto $P(U)$. □

We want to use Proposition 1.33 from [Hat01] from algebraic topology. This is also called the 'Lifting Lemma'.

Theorem 4.3 (Lifting of continuous maps)**.** *Suppose given a connected covering space* $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ *and a map* $f\colon (Y, y_0) \to (X, x_0)$ *with* Y *path-connected and locally path-connected. Then a lift* $\tilde{f}\colon (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ *of* f *exists if and only if* $f_*(\pi_1(Y, y_0)) \subseteq p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right)$.

The spaces in this theorem are *pointed spaces*. They are equipped with one distinguished point which is used to define the fundamental group π_1. π_1 is the group of equivalence classes of loops through this point. Loops are equivalent if and only if they are homotopic. The notation f_* denotes the induced homomorphism defined by $f_*([\alpha]) = [f \circ \alpha]$ where α is a loop in the domain of f and $[\alpha]$ is the homotopy equivalence class of α.

We use Theorem 4.3 to formulate a condition for the orientability of a continuous line field: we can check orientability on a selected number of loops. To define the spaces of line and unit vector fields we need a notation for restrictions of fields to submanifolds. Denote the space of sections of regularity R of the fiber bundle E on the submanifold $N \subset M$ as $\Gamma_R(E|_N)$.

Theorem 4.4 (Orientability of continuous line fields)**.** *Consider a continuous line field* $Q \in \Gamma_C(\mathcal{Q}^{\mathbb{S}} M)$*. Let* G *be a set of loops at a common point* p_0 *that generate the fundamental group* $\pi_1(M, p_0)$*.* Q *is orientable, i. e. there exists* $n \in \Gamma_C(\mathbb{S}M)$ *such that* $P(n) = Q$*, if and only if* Q *is orientable along every path* $\gamma \in G$*.*

We formulate the same in the language of Section 3.2 and allow fields that are not tangent to M.

Lemma 4.5 (Orientability of continuous line fields via embedding of M)**.** *Let* ι *be an isometric embedding of* M *into* $\mathbb{R}^N$ *as in Section 3.2 and let* $Q \in C(M, \mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N)$ *be a line field on but not necessarily tangent to* M*. Let* G *be a set of loops on* M *at a common point* p_0 *that generate the fundamental group* $\pi_1(M, p_0)$*. Then,* Q *is orientable, i. e. there exists* $n \in C(\mathbb{S}^{N-1})$ *such that* $P_N(n) = Q$*, if and only if* Q *is orientable along every path* $\gamma \in G$*.*

Theorem 4.4 and Lemma 4.5 are very similar. Therefore the proofs are combined. Parts in pink are for Theorem 4.4and parts in yellow are for Lemma 4.5. Additionally exchange $\mathbb{S}M$ with $\mathbb{S}^{N-1}$ and P with P_N for the proof of Lemma 4.5.

Proof. As Lemma 4.2 shows, $P\colon \mathbb{S}M \to \mathcal{Q}^{\mathbb{S}}M$$P_N\colon \mathbb{S}^{N-1} \to \mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N$ is a covering map. Use Q_{p_0} and an $n_0 \in \mathbb{S}M$ with $P(n_0) = Q_{p_0}$ as the distinguished points.

Connected manifolds are path-connected and locally path-connected.

For the 'if' part let $\gamma \in G$, i. e. $\gamma\colon [0,1] \to M$ continuous with $\gamma(0) = \gamma(1) = p_0$. Q is orientable along γ. That means that there exists $n \in$$\Gamma_C(\mathbb{S}M|_{\gamma([0,1])})$$C(\gamma([0,1]), \mathbb{S}^{N-1})$ with $P(n) = Q$. If the orientation of Q along this path has $n(\gamma(0)) = -n_0$, use $-n$ instead. Then, $n \circ \gamma$ is a continuous loop at n_0 in $\mathbb{S}M$ with $P_*[n \circ \gamma] = [P \circ n \circ \gamma] = [Q \circ \gamma] = Q_*[\gamma]$. Since γ is an arbitrary generator of $\pi_1(M, p_0)$ and Q_* and P_* are group homomorphisms, we get that $Q_*(\pi_1(M, p_0)) \subseteq P_*(\pi_1(\mathbb{S}M, n_0))$. Hence there exists $n \in C(M, \mathbb{S}M)$ with $P(n) = Q$ by Theorem 4.3, i. e. Q is orientable and $n \in \Gamma_C(\mathbb{S}M)$.

For the 'and only if' part restrict an orientation for Q on the paths in G. □

Corollary 4.6 (Orientability on simply connected manifolds)**.** *If* M *is simply connected every continuous line field* $Q \in \Gamma_C(\mathcal{Q}^{\mathbb{S}}M)$ *or* $Q \in C(M, \mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N)$ *is orientable.*

Proof. M being simply connected means that $\pi_1(M, p_0)$ is the trivial group for any point $p_0 \in M$. Hence it has no generators. Therefore the assumption in Theorem 4.4 or Lemma 4.5 is trivially fulfilled. □

Remark 4.7 (Topological restriction on the manifold). Hopf [Hop27] showed that the existence of unit vector fields is a topological property of the manifold. There exists a unit vector fields on a smooth, compact manifold without boundary if and only if the Euler characteristic is zero. This result is called the *Poincaré-Hopf Theorem*. For surfaces–two-dimensional manifolds–without boundary this implies that the only one with unit vector fields is the torus. For all other compact boundaryless surfaces the question of orientability of continuous line fields is uninteresting.

Example 4.8 (Sphere). The sphere $\mathbb{S}^2$ is simply connected but has Euler characteristic 2. Therefore every continuous line field is orientable but there exist no continuous unit vector fields and hence no continuous line fields.

On the 2-sphere the Poincaré-Hopf Theorem is called *Hairy Ball Theorem* [EG79].

Since the sphere and distorted spheres are a very common object of study and real-life application many applications circumvent this problem in some way. Usually the fields have *point defects* which are points on the sphere where the line or unit vector field is not well-defined or zero. For example Napoli and Vergori [NV12] study tiny droplets with liquid crystals on the surface. Since the liquid crystal layer is so thin in comparison to the size of the droplet it can be modeled as two-dimensional. Then, the liquid crystal must have defects that influence the properties of the entire droplet like bonds to neighboring particles.

Example 4.9 (Circle). The natural question that arises from Theorem 4.4 is if the converse is also true, i. e. if we can find a non-orientable line field on every manifold that is not simply connected.

In this simple form the converse is not true. To see this, consider the circle $\mathbb{S}^1$. $T_p\mathbb{S}^1$ is one-dimensional at every $p \in \mathbb{S}^1$. Therefore $\mathbb{S}\mathbb{S}^1$ consists of only two vectors at every base point and $\mathcal{Q}^{\mathbb{S}}\mathbb{S}^1$ even only of one element. Therefore there is only exactly one line field on the sphere and it is orientable.

We will also look at the one example of a boundaryless two-dimensional compact manifold with unit vector fields and construct a non-orientable line field: a torus. In order to show that a line field is indeed non-orientable we have to understand 'how it can fail'. Indeed, if we look at paths that are not loops but injective, every line field can be oriented. In the flat case this was done in Lemma 3 of [BZ11] and we will reduce the manifold case to this.

Proposition 4.10 (Orientability on a path). *Let $\gamma\colon [0,1] \to M$ be a continuous injective path on M and $Q \in \Gamma_C\left(\mathcal{Q}^{\mathbb{S}}M\big|_{\gamma([0,1])}\right)$ be a continuous line field along the path γ. Then, there exist exactly two orientations n^+ and $n^- \in \Gamma_C(\mathbb{S}M|_{\gamma([0,1])})$ such that*

$$Q_{\gamma(t)} = P\left(n^{\pm}_{\gamma(t)}\right) \text{ for all } t \in [0,1]$$

and $n^- = -n^+$. Equivalently, given either of the two possible initial orientations at $\gamma(0)$, there exists a unique continuous orientation with this initial orientation.

Proof. We use the parallel transport of γ to transform Q to a line field that maps from $[0,1]$ to $\mathcal{Q}^{\mathbb{S}}_{\gamma(0)}M$:

$$\tilde{Q}(t) := \operatorname{Par}^{\gamma}_{t0} Q_{\gamma(t)} \ .$$

For a definition of the parallel transport along γ from $\gamma(\tau)$ to $\gamma(t)$ see Theorem 4.32 in [Lee18]. Since P^{γ}_{t0} is a linear isometry, it maps $\mathcal{Q}^{\mathbb{S}}_{\gamma(t)}M$ to $\mathcal{Q}^{\mathbb{S}}_{\gamma(0)}M$. Hence $\tilde{Q}$ maps from $[0,1]$ to $\mathcal{Q}^{\mathbb{S}}_{\gamma(0)}M$. By choosing an orthonormal basis for $\mathcal{Q}^{\mathbb{S}}_{\gamma(0)}M$ we can identify it with the space $\mathcal{Q}$ as it is used by Ball and Zarnescu [BZ11]. Note that g written in an orthonormal basis is the identity matrix used in the definition of P used in [BZ11]. Furthermore the parallel transport $\mathrm{Par}^{\gamma}_{tt_0}$ commutes with the projection P which is clear when writing n and $P(n)$ in coordinates with respect to an orthonormal frame transported by $\mathrm{Par}^{\gamma}_{0t}$.

With this construction the statement follows directly from Lemma 3 in [BZ11]. Note that [BZ11] formally only considers the case $m = 3$ but the proof generalises trivially to any dimension. □

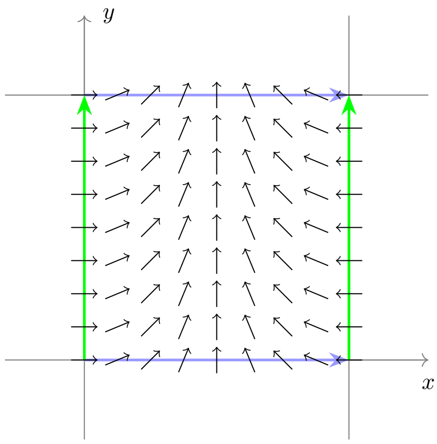


Figure 6: The (uncontinuous) unit vector field $n(x,y) = \cos\left(\frac{\pi}{2}x\right)\frac{\partial}{\partial x} + \sin\left(\frac{\pi}{2}x\right)\frac{\partial}{\partial y}$ on the flat torus $\mathbb{T}^2$. The colored edges of the square indicate which sides are identified to turn $[0,1]^2$ into $\mathbb{T}^2$.

Example 4.11 (Torus). There exist line and unit vector fields on the torus and it is not simply connected. Therefore it is a candidate surface to exhibit non-orientable line fields. In order to construct a non-orientable line field we take a loop that is not contractable, for example a circle around the hole. Call it $\gamma\colon [0,1] \to M$. We know by Proposition 4.10 that any line field can be oriented along a path. Hence on $\gamma([0,1))$ we can find an orientation and thus the problem must occur at $\gamma(0) = \gamma(1)$ and Q must somehow force the oriented version n to 'turn around' along the curve. After defining Q suitably on this loop we also have to check if we can extend it to all of M.

For simple computation use the square $[-1,1]^2$ with opposite sides identified as a model for the torus. Define the discontinuous unit vector field $n(x,y) := \cos\left(\frac{\pi}{2}x\right)\frac{\partial}{\partial x} + \sin\left(\frac{\pi}{2}x\right)\frac{\partial}{\partial y}$. It is continuous on $(-1,1)^2$ and $n(0,y) = -n(1,y)$ and $n(x,0) = n(x,1)$ for all $x, y \in [0,1]$. Hence $Q := P(n)$ is continuous on the entire torus but on the path $\gamma\colon [0,1] \to M$ with $\gamma(t) = (t,0)$ an orientation of Q it must be equal to n (or $-n$) on $(0,1)$ but then is discontinuous at $\gamma(0) = \gamma(1)$. In Figure 6 γ can be any path parallel to the x-axis.

5 Orientability of Sobolev line fields

Since algebraic topology is only concerned with continuous maps, it has no tools to directly study Sobolev line fields. Instead we use approximation results that help to reduce the orientability question to continuous fields. Smooth functions between manifolds are in general not dense in the Sobolev function space, though. Therefore we will use two specialised density results for the simply-connected case (Theorem 5.1 in Section 5.1) and for surfaces (Theorem 5.11 in Section 5.2).

5.1 Orientability on simply-connected manifolds

If M is simply-connected, smooth fields are in general not dense in the norm sense but in a weak sense as Pakzad and Rivière [PR03] showed.

Theorem 5.1 (Sequentially weak density of smooth maps between manifolds)**.** *([Theorem I PR03, p. 225]) Let M, N be compact smooth manifolds with M simply connected. The Sobolev space $W^{1,2}(M,N)$ is defined as $\{f \in W^{1,2}(M,\mathbb{R}^d) \,|\, f \in N \text{ a. e.}\}$ where N is embedded isometrically into $\mathbb{R}^d$. The norm is inherited from $W^{1,2}(M,\mathbb{R}^d)$. Then for $u \in W^{1,2}(M,N)$, there exists a sequence $(u^{(k)})_k$ with $u^{(k)} \in C^\infty(M,N)$ for all $k \in \mathbb{N}$, such that $u^{(k)}$ converges weakly to u.*

To use Theorem 5.1 we need to show that the weak limit of orientable continuous approximations is also orientable.

Proposition 5.2 (Orientability preserved by weak convergence)**.** *Let $q \in [1,\infty)$ and*

$$\begin{aligned} Q_{(k)} &\in W^{1,q}(M, \mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N) && \text{for } k \in \mathbb{N}, \\ Q_{(k)} &\rightharpoonup Q \in W^{1,q}(M, \mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N) && \text{as } k \to \infty, \\ \text{and } Q_{(k)} &= P_N(n_{(k)}) \text{ with } n_{(k)} \in W^{1,q}(M, \mathbb{S}^{N-1}) && \text{for } k \in \mathbb{N}. \end{aligned}$$

Then there exists $n \in W^{1,q}(M, \mathbb{S}^{N-1})$ with $P_N(n) = Q(n)$.

If additionally $\iota\colon M \hookrightarrow \mathbb{R}^N$ is an isometric embedding of M as in Section 3.2, and $Q \in \Gamma_{W^{1,q}}(\mathcal{Q}^{\mathbb{S}'}M)$ is tangent to M, then $n \in \Gamma_{W^{1,q}}(\mathbb{S}M)$ is also tangent to M.

Proof. In order to get a weak limit of the $n_{(k)}$'s we need to show that the sequence is bounded. For this, calculate the Euclidean derivative $\overline{\nabla}_v \hat{n}$ for $\hat{n}$ weakly differentiable, a direction $v \in TM$ in terms of $\hat{Q} = P_N(\hat{n})$. Note that in $\mathbb{R}^N$ the musical operators $\sharp$ and $\flat$ are given by transposition $\cdot^T$. The vector $\hat{n}$ and the contravariant tensor $\hat{Q}$ are here interpreted as (bi)linear maps on one and two covectors, respectively.

As a preparation, note that $\hat{n} \cdot \hat{n} = |\hat{n}|^2 = 1$ and therefore,

$$0 = \frac{1}{2}\overline{\nabla}_v(\hat{n}\cdot\hat{n}) = \frac{1}{2}(\overline{\nabla}_v\hat{n})\cdot\hat{n} + \frac{1}{2}\hat{n}\cdot(\overline{\nabla}_v\hat{n}) = \overline{\nabla}_v\hat{n}\cdot\hat{n} = \overline{\nabla}_v\hat{n}(\hat{n}^T) \ . \tag{27}$$

Then,

$$\begin{aligned} \overline{\nabla}_v\hat{Q}(\hat{n}^T,\cdot) &= \overline{\nabla}_v\left(\hat{n}\otimes\hat{n}\right)\left(\hat{n}^T\right) \\ &= (\overline{\nabla}_v\hat{n})(\hat{n}^T)\otimes\hat{n} + \hat{n}(\hat{n}^T)\otimes\overline{\nabla}_v\hat{n} \\ &= \overline{\nabla}_v\hat{n} && ((27) \text{ and } \hat{n}(\hat{n}^T) = \hat{n}\cdot\hat{n} = 1) \ . \end{aligned} \tag{28}$$

Hence, $\overline{\nabla}\hat{n}$ is in L^q. Since also $|\hat{n}| = 1$ and M is compact, $\hat{n} \in W^{1,q}(M, \mathbb{S}^{N-1})$ and the norm of $\hat{n}$ is bounded by the norm of $\hat{Q}$.

Since $(Q_{(k)})_k$ converges weakly, it is a bounded sequence by the Uniform Boundedness Principle. Therefore, the previous calculation (28) shows that $(n_{(k)})_k$ is bounded as well. The Banach-Alaoglu Theorem further gives a subsequence $(n_{(k_l)})_l$ of $(n_{(k)})_k$ that weakly converges to some $n \in W^{1,q}(M, \mathbb{R}^N)$. By the Kondrakov Theorem [§11 2.34 Theorem Aub82] $W^{1,q}(M, \mathbb{R}^N)$ is compactly embedded in $L^q(M, \mathbb{R}^N)$. Therefore, we can find a further subsequence $(n_{(k_s)})_s$ of $(n_{(k_l)})_l$ that converges in L^q-norm and therefore pointwise a. e.. The same reason shows that $(Q_{(k_s)})_s$ has a subsequence that converges pointwise a. e. and hence $P_N(n) = Q$, i. e. Q is orientable.

It remains to be shown that n is tangent to M if Q is tangent to M. For the sake of contradiction, assume $n_p \cdot \eta = n_p(\eta^T) \neq 0$ for some $p \in M$ and $\eta \perp T_pM$. Then

$$P_N(n)(\eta^T, \eta^T) = Q(\eta^T, \eta^T) = (n \otimes n)(\eta^T, \eta^T) = (n(\eta^T))^2 \neq 0 \ .$$

If Q is tangent to M, i. e. $Q \in \Gamma_{W^{1,q}}(\mathcal{Q}^{\mathbb{S}'}M)$, this is false a. e.. Hence $n \in \Gamma_{W^{1,q}}(\mathbb{S}M)$ is a. e. tangent to M. □

Remark 5.3 (Choice of codomain). If we only used the intrinsic definition of the tensor fields it would be unclear how to use Theorem 5.1. If we used it with $N = \mathcal{Q}^{\mathbb{S}}M$, we could not guarantee that it holds for the approximations that $(Q_{(k)})_p \in T_pM$. If we used it with $N = \mathcal{Q}^{\mathbb{S}}_pM$ we have to somehow identify all tangent spaces which is–in the general case–not possible in a smooth way. By embedding M and allowing non-tangent approximations, we solve this problem but need Corollary 4.6 and Proposition 5.2 for $\mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N$- and $\mathbb{S}^{N-1}$-valued functions.

Theorem 5.4 (Sobolev orientability on simply connected manifolds)**.** *If M is simply-connected, every Sobolev line field $Q \in \Gamma_{W^{\nabla,q}}(\mathcal{Q}^{\mathbb{S}}M)$ is orientable for $q \in [2, \infty)$, i. e. there exists $n \in \Gamma_{W^{\nabla,q}}(\mathbb{S}M)$ with $P(n) = Q$.*

Proof. By the Nash embedding theorem [Nas56] we can embed the base manifold M isometrically into $\mathbb{R}^N$ for some $N \in \mathbb{N}$. As usual we call this embedding ι.

Since M is compact and $q \geq 2$, the line field Q is also in the Sobolev space with exponent 2: $Q \in \Gamma_{W^{\nabla,2}}(\mathcal{Q}^{\mathbb{S}}M)$. By Lemma 3.13 we can view Q as a map into the surrounding Euclidean space: $\iota_Q(Q) \in W^{1,2}(M, \mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N)$. The codomain $\mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N$ is a model for $\mathbb{RP}^{N-1}$ as mentioned in Section 4 and hence a smooth compact manifold. Therefore, by Theorem 5.1, there exists a sequence $(Q_{(k)})_k$ in $C^\infty(M, \mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N)$ converging weakly to $\iota_Q(Q)$. By Corollary 4.6, $Q_{(k)}$ is orientable for all $k \in \mathbb{N}$ with $P(n_{(k)}) = Q_{(k)}$ and $n_{(k)} \in C(M, \mathbb{S}^{N-1})$.

By Lemma 3.11 the orientations $n_{(k)} \in W^{1,2}(M, \mathbb{S}^{N-1})$ are also weakly differentiable.

Now we apply Proposition 5.2. It shows that $\iota_Q(Q)$ is orientable with $P_N(n) = \iota_Q(Q)$ and $n \in W^{1,2}(\mathbb{S}M)$. The calculation (28) gives a formula for ∇n that shows that ∇n is also in L^q since $\nabla \iota_*(Q)$ is in L^q. Transferring n back with ι_*^{-1}, we get $P(\iota_*^{-1}n) = Q$, i. e. Q is orientable. □

5.2 Orientability on surfaces

As mentioned in the introduction liquid crystals exist in some cases like [Keb+14] as very thin films that are best modelled as two-dimensional surfaces. Therefore, the case $\dim M = 2$ is of special interest. Section 4 of [BZ11] gives a roadmap how to transfer the orientability criterion from continuous to Sobolev line fields. In order to use those ideas, we use that the tangent bundle is trivial in the case of oriented surfaces with unit vector fields. This allows us to consider vector fields as maps into $\mathbb{R}^2$ as discussed in Section 3.3. Unorientable surfaces are shortly discussed in the outlook 7.

Lemma 5.5 (Triviality of two dimensions)**.** *Let M be a compact orientable surface. Then, either M admits no smooth unit tangent field or has a global orthonormal frame.*

The prominent example of a manifold without unit tangent field is the sphere by the Hairy Ball theorem [EG79]. A manifold with a global frame is called *parallelizable* and its tangent bundle is called *trivial*. Any global frame can be orthonormalized to an orthonormal global frame with the Gram-Schmidt process.

Proof. If M admits no smooth unit tangent field, we are done. Otherwise take some $X \in \Gamma_{C^\infty}(\mathbb{S}M)$ and choose an orientation for M. Then, in each point $p \in M$, the orthogonal complement $X_p^\perp$ is a one-dimensional subspace. It contains exactly two unit vectors. Exactly one of them forms a positively oriented basis of T_pM with X_p, call it Y_p. Since X is smooth and M is orientable, Y is also smooth. □

Similarly to Section 3.3, we assume throughout this subsection that M is a compact orientable surface and choose a global smooth orthonormal frame $(B_1, B_2) \in (\Gamma_{C^\infty}(\mathbb{S}M))^2$. We write $W^{1,2}_{\shortparallel}(\mathbb{S}M)$ and $W^{1,2}_{\shortparallel}(\mathcal{Q}^{\mathbb{S}}M)$ but everything holds for $\Gamma_{W^{\nabla,2}}(\mathbb{S}M)$ and $\Gamma_{W^{\nabla,2}}(\mathcal{Q}^{\mathbb{S}}M)$ as well since the spaces are isomorphic as shown in Lemma 3.16. Note that $|n|_g = 1$ is equivalent to $\left|\left(\langle B_1, n\rangle_g, \langle B_2, n\rangle_g\right)^T\right| = 1$ since (B_1, B_2) is orthonormal.

The goal is to show that orientability of a line field on a surfaces is equivalent to orientability of the line field on a set of loops that generate the fundamental group. While Section 5.2.1 shows that an orientable line field on the entire surface is orientable on the generators, Section 5.2.2 shows the opposite direction.

5.2.1 Orientability from the surface to loops

The easier direction is to show that an orientable line field is also orientable on loops. Indeed, this is possible in any dimension and for non-trivial tangent bundles but this would require yet another way of defining Sobolev vector fields that is compatible with fractional Sobolev vector fields and the trace theorem. It would extend this thesis more than it would give insight.

The continuous case is trivial since the chosen orientation can simply be restricted to the loop. In the Sobolev case the trace is defined via approximation and hence it needs to be shown that orientability is stable under this approximation. For this we need to show that the projection P is continuous.

Lemma 5.6 (Continuity of projection on surface)**.** *For $q \in [1, \infty)$,*

$$P\colon \left\{ n \in W^{1,q}_{\shortparallel}(TM) \,\middle|\, |n|_g \le 1 \right\} \to W^{1,q}_{\shortparallel}(\mathcal{Q}M)$$

is well-defined and continuous. Here P is defined on a larger domain but in the usual way $(P(n))_p := s\left(n_p \otimes n_p - \frac{1}{2} g_p^{\sharp\sharp}\right)$ for $p \in M$.

Proof. $\mathcal{Q}M$ is defined just such that $P(n)$ maps into $\mathcal{Q}M$. To show that $P(n) \in W^{1,q}_{\shortparallel}(\mathcal{Q}M)$ and that P is continuous, calculate $\|P(n)\|^q_{W^{1,q}_{\shortparallel}}$ in terms of $\|n\|^q_{W^{1,q}_{\shortparallel}}$. To simplify notation, let $n^i := \langle B_i, n \rangle_g$ and $Q := P(n)$ with $Q^{ij} := \left\langle B_i \otimes B_j, Q \right\rangle_g$ for $i, j \in \{1, 2\}$.

$$
\begin{aligned}
Q^{ij} &= \left\langle B_i \otimes B_j, s(n \otimes n - \frac{1}{2} g^{\sharp\sharp}) \right\rangle_g \\
&= s \langle B_i, n \rangle_g \left\langle B_j, n \right\rangle_g - \frac{s}{2} g(B_i, B_j) = s n^i n^j - \frac{s}{2} \delta^{ij} \\
\nabla Q^{ij} &= s \left((\nabla n^i) n^j + n^i (\nabla n^j) \right)
\end{aligned}
$$

Since $|n|_g$ is bounded by 1, it follows that $\|Q\|_{W^{1,q}_{\shortparallel}}$ is finite, so P is well-defined. Now we estimate the difference $Q - \tilde{Q} := P(n) - P(\tilde{n})$ for $n, \tilde{n} \in W^{1,q}_{\shortparallel}(\mathbb{S}M)$.

$$
\begin{aligned}
Q^{ij} - \tilde{Q}^{ij} &= s \left(n^i n^j - \frac{1}{2} \delta^{ij} - \tilde{n}^i \tilde{n}^j + \frac{1}{2} \delta^{ij} \right) \\
&= s(\underbrace{(n^i - \tilde{n}^i)}_{\to 0} \underbrace{n^j}_{\leq 1} + \underbrace{(n^j - \tilde{n}^j)}_{\to 0} \underbrace{\tilde{n}^i}_{\leq 1}) \to 0 \text{ in } L^q \\
\nabla Q^{ij} - \nabla \tilde{Q}^{ij} &= s \left((\nabla n^i) n^j + n^i (\nabla n^j) - (\nabla \tilde{n}^i) \tilde{n}^j - \tilde{n}^i (\nabla \tilde{n}^j) \right) \\
&= s(\underbrace{(\nabla n^i - \nabla \tilde{n}^i)}_{\to 0} \underbrace{n^j}_{\leq 1} + \underbrace{(\nabla n^j - \nabla \tilde{n}^j)}_{\to 0} \underbrace{\tilde{n}^i}_{\leq 1} \\
&\qquad + \underbrace{2(\nabla n^j)}_{\in L^q} \underbrace{(n^i - \tilde{n}^i)}_{\to 0} + \underbrace{(\nabla \tilde{n}^i - \nabla n^i)}_{\to 0} \underbrace{(n^j - \tilde{n}^j)}_{\to 0}) \\
&\to 0 \text{ in } L^q \text{ as } \tilde{n} \to n \text{ in } W^{1,q}_{\shortparallel}
\end{aligned}
$$

The second to last term goes to zero by the dominated convergence theorem with $\left|4 \nabla n^j\right|$ as the dominating integrable function. We see that $\|P(n) - P(\tilde{n})\|_{W^{1,q}_{\shortparallel}}$ goes to zero if $\|n - \tilde{n}\|_{W^{1,q}_{\shortparallel}}$ goes to zero. Hence P is continuous. $\square$

Proposition 5.7 (Orientability on loops)**.** *Let $N \subset M$ be a 1-dimensional submanifold of M and let $Q \in W^{1,2}_{\shortparallel}(\mathcal{Q}^{\mathbb{S}}M)$ be orientable with $P(n) = Q$ and $n \in W^{1,2}_{\shortparallel}(\mathbb{S}M)$. Then,* $\operatorname{Tr}_N Q$ *is orientable with* $P(\operatorname{Tr}_N n) = \operatorname{Tr}_N Q \in W^{\frac{1}{2},2}(\mathcal{Q}^{\mathbb{S}}M|_N)$.

Proof. The Meyers-Serrin density result Theorem 3.8 gives an approximating sequence $n_{(k)} \in C^\infty_{\shortparallel}(TM)$ with $\left|n_{(k)}\right|_g \leq \|n\|_{L^\infty} = 1$ that converges in $W^{1,2}$ to n. We showed that P is continuous in the previous Lemma 5.6. and thus $P(n_{(k)}) \to P(n) = Q$. Since the $n_{(k)}$'s are smooth, the trace operator on the $n_{(k)}$'s is just restriction and thus trivially commutes with P:

$$
P(\operatorname{Tr}_N n_{(k)}) = \operatorname{Tr}_N P(n_{(k)}) \to \operatorname{Tr}_N \lim_{k \to \infty} P(n_{(k)}) = \operatorname{Tr}_N Q \text{ in } W^{\frac{1}{2},2}(\mathcal{Q}M|_N) \,.
$$

The right hand side $\operatorname{Tr}_N P(n_{(k)})$ converges because Tr_N is continuous, see Theorem 3.20. On the left hand side we notice that $\operatorname{Tr}_N n_{(k)}$ are also $L^2(TM|_N)$-functions. Tr_N is also continuous as an operator from $W^{1,2}$ to L^2 and thus $\operatorname{Tr}_N n_{(k)}$ converges to $\operatorname{Tr}_N n$. Recalling the previous proof of Lemma 5.6 we see that P is also continuous on L^2 and hence $P(\operatorname{Tr}_N n_{(k)}) \to P(\operatorname{Tr}_N n)$ in L^2. This shows $P(\operatorname{Tr}_N n) = \operatorname{Tr}_N Q$.

In order to show that Q is orientable, we still need to show that $\operatorname{Tr}_N n \in W^{1,2}_{\shortparallel}(\mathbb{S}M)$, i. e. $|n| = 1$ a. e. on M. We use Theorem 5.7 of [EG15]. It states that

$$
\lim_{r \to 0} \frac{1}{|B(x,r) \cap M|} \int_{B(x,r) \cap M} |n(y) - \operatorname{Tr}_N n(x)| \, \mathrm{d}y = 0 \text{ for almost all } x \in N \,.
$$

It is proven for Euclidean domains but since it is a local property it also holds on manifolds. Now

$$\begin{aligned} 1-|\operatorname{Tr}_N n(x)| &= \frac{1}{|B(x,r)\cap M|}\int_{B(x,r)\cap M} |1-|\operatorname{Tr}_N n(x)||\,\mathrm{d}y \\ &= \frac{1}{|B(x,r)\cap M|}\int_{B(x,r)\cap M} ||n(y)|-|\operatorname{Tr}_N n(x)||\,\mathrm{d}y \\ &\le \frac{1}{|B(x,r)\cap M|}\int_{B(x,r)\cap M} |n(y)-\operatorname{Tr}_N n(x)|\,\mathrm{d}y \qquad \to 0 \quad (\text{a.e. as } r\to 0) \end{aligned}$$

and hence $|\operatorname{Tr}_N n| = 1$ a. e. on N. □

5.2.2 Orientability from loops to the surface

We can to view the 'squaring' $n\otimes n$ in the projection $Q = P(n) = s\left(n\otimes n - \frac{1}{2}g\right)$ as an actual squaring operation by identifying the codomain $\mathbb{S}^1$ with the unit circle in the complex plane $\mathbb{C}$.

Definition 5.8 (Complex version of line field)**.** We define the following auxiliary complex valued functions for $Q\in W^{1,2}_{\shortparallel}(\mathcal{Q}^{\mathbb{S}}M)$ and $n\in W^{1,2}_{\shortparallel}(\mathbb{S}M)$ with the coordinates $Q^{ij} := \left\langle B_i\otimes B_j, Q\right\rangle_g$ and $n^i := \langle B_i, n\rangle_g$ $(i,j\in\{1,2\})$

$$\begin{aligned} A(Q) &:= \frac{2}{s}\left(\langle B_1\otimes B_1, Q\rangle_g + \mathrm{i}\,\langle B_1\otimes B_2, Q\rangle_g\right) = \frac{2}{s}\left(Q^{11}+\mathrm{i}Q^{12}\right)\in\mathbb{S}^1\subset\mathbb{C} \\ Z(n) &:= \langle B_1, n\rangle_g + \mathrm{i}\,\langle B_2, n\rangle_g = n^1+\mathrm{i}n^2\in\mathbb{S}^1\subset\mathbb{C}\ . \end{aligned}$$ ◀

Remark 5.9 (Squaring). $A(Q)$ is defined in a way that we have $A(P(n)) = Z(n)^2$. To show this, let $n\colon M\to\mathbb{S}M$ and note that $g^{\sharp\sharp} = B_1\otimes B_1 + B_2\otimes B_2$ since (B_1,B_2) is orthonormal. Then

$$\begin{aligned} A(P(n)) &= A\left(s\left(n\otimes n-\frac{1}{2}g\right)\right) \\ &= \frac{2}{s}\left(\left\langle B_1\otimes B_1, s\left(n\otimes n-\frac{1}{2}g^{\sharp\sharp}\right)\right\rangle_g + \mathrm{i}\left\langle B_1\otimes B_2, s\left(n\otimes n-\frac{1}{2}g^{\sharp\sharp}\right)\right\rangle_g\right) \\ &= 2n^1\cdot n^1 - \frac{2}{2}\left\langle B_1\otimes B_1, g^{\sharp\sharp}\right\rangle_g + 2\mathrm{i}n^1\cdot n^2 - \frac{2}{2}\mathrm{i}\left\langle B_1\otimes B_2, g^{\sharp\sharp}\right\rangle_g \\ &= (n^1)^2 - (1-(n^1)^2) + 2\mathrm{i}n^1n^2 \\ &= (n^1)^2 - (n^2)^2 + 2\mathrm{i}n^1n^2 \qquad\qquad (|n|_g^2 = (n^1)^2+(n^2)^2 = 1) \\ &= (n^1+\mathrm{i}n^2)^2 = Z(n)^2\ . \end{aligned}$$

Since $\mathcal{Q}M$ only consists of symmetric and trace-free tensors, $A(Q)$ determines Q completely, so A is a $\mathbb{R}$-linear bijection.

Ball and Zarnescu [BZ11] show that for flat two-dimensional domains with holes we can check if a line field Q is orientable by checking if the winding numbers of $A(Q)$ on the boundaries are even. The winding number describes how often a function wraps around the circle along a loop. If this is even you can find a square root $Z(n)$ that wraps around half as often. From [Hir97, pp. 120–130] and [Theorem A.3 dMGP91] we get that there exists a *winding number* for $f \in W^{\frac{1}{2},2}(\mathbb{S}^1, \mathbb{S}^1)$ (the codomain $\mathbb{S}^1$ as a subset of $\mathbb{C}$) computed by

$$\operatorname{wind} f = \frac{1}{2\pi i} \int_{\mathbb{S}^1} f^{-1} \frac{\partial f}{\partial \theta} \, \mathrm{d}\theta \in \mathbb{Z} \ . \tag{29}$$

These papers call the winding number 'degree' since it is special case of a more general concept from differential geometry that also allows more than one dimension. But since in our one-dimensional case the term 'winding number' is more self-explanatory we use it here.

The integral in (29) is to be understood in the sense of distributions since $f, f^{-1} \in W^{\frac{1}{2},2}(\mathbb{S}^1, \mathbb{S}^1)$ and $\frac{\partial f}{\partial \theta} \in W^{-\frac{1}{2},2}(\mathbb{S}^1, \mathbb{S}^1)$. The winding number is is an integer and it is invariant under sufficiently small perturbations of f. It is also invariant under reparametrisation and thus allows any loop as the domain. For some more discussion on the degree, see Section 4 of [BZ11].

As mentioned above, a line field is orientable on a hole boundary, which is a loop, if and only if the degree of $A(Q)$ is even, as shown in Proposition 6 in [BZ11]. The same proof can be used for our case.

Proposition 5.10 (Orientability on loops)**.** *Let* $Q \in W^{1,2}_{\shortparallel}(\mathcal{Q}^{\mathbb{S}}M)$ *and let* $\gamma\colon \mathbb{S}^1 \to M$ *be a loop on* M*. Then* $\operatorname{Tr} Q := \operatorname{Tr}_{\gamma(\mathbb{S}^1)} Q \in W^{\frac{1}{2},2}_{\shortparallel}(\mathcal{Q}^{\mathbb{S}}M|_{\gamma(\mathbb{S}^1)})$ *is orientable if and only if* $\operatorname{wind}(A(\operatorname{Tr} Q)) \in 2\mathbb{Z}$*. Moreover if it is orientable with* $n \in W^{\frac{1}{2},2}_{\shortparallel}(\mathbb{S}M|_{\gamma(\mathbb{S}^1)})$ *with* $\operatorname{Tr} Q = P(n)$*, then* $2\operatorname{wind}(n) = \operatorname{wind}(\operatorname{Tr} Q)$*.*

Proof. $(\operatorname{Tr}_{\gamma(\mathbb{S}^1)} A(Q))\circ\gamma$ and $(\operatorname{Tr}_{\gamma(\mathbb{S}^1)} Z(n))\circ\gamma$ are $W^{\frac{1}{2},2}(\mathbb{S}^1, \mathbb{S}^1)$-functions by Theorem 3.20. Therefore, the same proof as for Proposition 6 in Ball and Zarnescu [BZ11] holds. ☐

For continuous line fields we already know that orientability can be checked on a set of loops that generate the fundamental group. In order to use this for Sobolev line fields we approximate them with smooth maps.

Theorem 5.11 (Density of smooth maps on surfaces)**.** *(from [Proposition in section 4 SU83]) Let* M *be a 2-dimensional compact manifold. Let* N *be a compact manifold without boundary. Then,* $C^\infty(M, N)$ *is dense in* $W^{1,2}(M, N)$*.*

Note that the authors denote $W^{1,2}$ by L^2_1. As the authors show, this is in general not true if M is of higher dimension.

Now we are ready to prove how to characterize orientable Sobolev line fields on surfaces. The ideas for the proof stem from Proposition 7 of [BZ11].

Theorem 5.12 (Orientability of Sobolev line fields on surfaces)**.** *Let* $Q \in W^{1,2}_{\shortparallel}(\mathcal{Q}^{\mathbb{S}}M)$ *where* M *is a Riemannian orientable surface such that there exists a smooth tangent unit vector field on* M*. Choose a finite set of loops* G *at a common point* p_0 *that generate the fundamental group* $\pi_1(M, p_0)$*. Then,* Q *is orientable if and only if* $\operatorname{wind}\left(A\left(\operatorname{Tr}_{\gamma(\mathbb{S}^1)} Q\right)\right) \in 2\mathbb{Z}$ *for all* $\gamma\colon \mathbb{S}^1 \to M$ *in* G*.*

Remark 5.13 (Fundamental group is finitely generated)*.* Theorem 5.12 assumes that the fundamental group $\pi_1(M, p_0)$ of a surface is finitely generated. For boundaryless orientable compact surfaces this is shown by section 17c of [Ful97] with the standard form for those surfaces. For surfaces with boundaries a proof outline is given by MartianInvader [Mar14]:

> Differentiable manifolds can always be given the structure of PL [piece-wise linear] manifolds, which can be triangulated into simplicial complexes. By shrinking a spanning tree of the 1-skeleton of this simplicial complex, we can obtain a CW complex X with a single 0-cell. This complex is no longer a manifold, but has the same fundamental group as the original manifold, since quotiening out by a contractible subspace is a homotopy equivalence.
>
> If the manifold is compact, it has a simplicial decomposition with a finite number of cells. This carries over to X. But the fundamental group of a CW complex with a single 0-cell has a presentation with a generator for each 1-cell and a relation for each 2-cell. Thus X, and therefore the original manifold, has a finitely presented fundamental group.

Proof. The first direction is proven in the previous section by Proposition 5.7 together with Proposition 5.7. For the opposite direction let $Q \in W^{1,2}_{\shortparallel}(\mathcal{Q}^{\mathbb{S}}M)$ with the given winding number conditions. Approximate Q with a sequence $Q_{(k)}$ in $C^1(\mathcal{Q}^{\mathbb{S}}M)$ in $W^{1,2}_{\shortparallel}(\mathcal{Q}^{\mathbb{S}}M)$ by Theorem 5.11. We will show that $Q_{(k)}$ is orientable for sufficiently large k. Take any loop $\gamma \in G$. The winding number of Q along this loop is even by assumption: $\operatorname{wind}\left(A\left(\operatorname{Tr}_{\gamma(\mathbb{S}^1)} Q\right)\right) \in 2\mathbb{Z}$. Since the Tr operator is continuous by Theorem 3.20 $\left\|A(\operatorname{Tr}_{\gamma(\mathbb{S}^1)} Q) - A(\operatorname{Tr}_{\gamma(\mathbb{S}^1)} Q_{(k)})\right\|_{W^{\frac{1}{2},2}}$ tends to 0 as $k \to \infty$. Now we use that the winding number is stable under small perturbations. For this we employ the following result.:

Theorem 5.14 (BMO continuity of the degree)**.** *(Theorem 1 from [Theorem 1 BN95]) Let $u \in \mathrm{VMO}(X,Y)$ where X and Y are smooth compact manifolds without boundaries. Then there exists $\delta > 0$ depending on u, such that if $v \in \mathrm{VMO}(X,Y)$ and $\operatorname{dist}(u,v) < \delta$, then $\deg(v) = \deg(u)$.*

We use this theorem with $X = Y = \mathbb{S}^1$ and deg = wind as mentioned above. The space VMO is the space of measurable functions of *vanishing mean oscillations.* Its norm is the BMO-norm (*bounded mean oscillation*). The important information for us is that VMO is a superspace of $W^{\frac{1}{2},2}$ with continuous injection. This is shown in Example 2, case 2 in the same work [BN95]. That means that $\left\|A(\operatorname{Tr}_{\gamma(\mathbb{S}^1)} Q) \circ \gamma\right\|_{\mathrm{BMO}} < \infty$ and there is some $C > 0$ such that for all $k \in \mathbb{N}$

$$\begin{aligned}&\left\|A(\operatorname{Tr}_{\gamma(\mathbb{S}^1)} Q) \circ \gamma - A(\operatorname{Tr}_{\gamma(\mathbb{S}^1)} Q_{(k)}) \circ \gamma\right\|_{\mathrm{BMO}} \\ &\qquad \leq C\left\|A(\operatorname{Tr}_{\gamma(\mathbb{S}^1)} Q) \circ \gamma - A(\operatorname{Tr}_{\gamma(\mathbb{S}^1)} Q_{(k)}) \circ \gamma\right\|_{W^{\frac{1}{2},2}} \to 0, \quad k \to \infty \ .\end{aligned}$$

Hence $\operatorname{wind}(A(\operatorname{Tr}_{\gamma(\mathbb{S}^1)} Q_{(k)})) = \operatorname{wind}(A(\operatorname{Tr}_{\gamma(\mathbb{S}^1)} Q)) \in 2\mathbb{Z}$ for all sufficiently large k. By Proposition 5.10 this means that $\operatorname{Tr}_{\gamma(\mathbb{S}^1)} Q_{(k)}$ is orientable in $W^{\frac{1}{2},2}$. In order to use the theory for continuous line fields we need to show that $\operatorname{Tr}_{\gamma(\mathbb{S}^1)} Q_{(k)}$ is also orientable in the class of continuous unit vector fields. This is shown by [BZ11] together with [dMGP91]. Recall that $Q_{(k)}$ is smooth by definition.

Theorem 5.15 (Continuous and $W^{\frac{1}{2},2}$ orientability on the line)**.** *([Lemma 9 BZ11]) Let $I \subset \mathbb{R}$ be an open set. Take $f\colon I \to \mathbb{R}$ be such that $f \in W^{\frac{1}{2},2}(I,\mathbb{R})$ and $f^2 \in C(I,\mathbb{R})$. Then, there exists $f^* \in C(I,\mathbb{R})$ so that $f^* = f$ almost everywhere on I.*

Theorem 5.16 (Angle function)**.** *([Theorem A.3 dMGP91]) Let f be a function of Sobolev class $W^{\frac{1}{2},2}$ from the circle $\mathbb{S}^1$ to itself. Then, there exists an integer n and* *a real function $g \in W^{\frac{1}{2},2}$ on $\mathbb{S}^1$, unique up to an integral multiple of 2π, such that $f = z^n \exp(\mathrm{i}g)$. The winding number n is given by $n = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{S}^1} f^{-1} \frac{\partial f}{\partial x} \,\mathrm{d}x$.*

Since G is finite, there is some $k_0 \in \mathbb{N}$ such that for all $k > k_0$ and all $\gamma \in G$, $\mathrm{Tr}_{\gamma(\mathbb{S}^1)} Q_{(k)}$ is orientable in $W^{\frac{1}{2},2}$. At this point we apply Theorem 4.4 to see that those $Q_{(k)}$ are orientable with $n_{(k)} \in C(\mathbb{S}M)$ such that $P(n_{(k)}) = Q_{(k)}$. Lemma 3.11 shows that the $n_{(k)}$'s are in $W^{\frac{1}{2},2}$. This is necessary to find a limit of the $n_{(k)}$'s. Lemma 3.11 was intended for fields into a surrounding Euclidean space but the statement holds for $N = 2 = m$ as well. The lemma uses the projection operator P_N, but as defined in Definition 3.9 P differs from P_N merely by the affine linear transformation $Q_p \mapsto \frac{Q_p}{s} + \frac{1}{m} g^{\sharp\sharp}$ with the smooth and bounded $g^{\sharp\sharp}$. So the result is also applicable here.

Now we have a sequence of line fields that converges in norm and which elements are orientable in $W^{1,2}$. This is a stronger statement than in the previous Section 5.1 where we dealt with weak convergence. So it natural to expect that the orientations $n_{(k)}$ also converge in norm. However, this is not the case since there are always two possible orientations for each $Q_{(k)}$ and hence the $n_{(k)}$'s can 'swap back and forth' instead of converging.

Therefore, we employ Proposition 5.2 again to show that a subsequence of the $n_{(k)}$'s converge weakly. Proposition 5.2 is mainly intended for fields on a manifold into a surrounding Euclidean space but it is also valid for $\mathcal{Q}^{\mathbb{S}'}\mathbb{R}^N$ for any $N \in \mathbb{N}$. Instead of $\mathcal{Q}^{\mathbb{S}'}\mathbb{R}^2$, the line fields here map into $\mathcal{Q}^{\mathbb{S}}\mathbb{R}^2$. Those two spaces are related via the affine transformation $Q \mapsto \frac{Q}{s} + \frac{1}{m} g^{\sharp\sharp}$ and therefore Proposition 5.2 is also applicable here. Hence, there exists $\tilde{n} \in \Gamma_{\shortparallel}^{W^{1,2}}(\mathbb{S}M)$ with $P_N(\tilde{n}) = \frac{Q}{s} + \frac{1}{m} g^{\sharp\sharp}$ and thus $P(n) = Q$ and Q is orientable. □

6 Orientability of minimizers of the harmonic energy

Models for liquid crystals should not only describe the matter, but also predict how it reacts under conditions enforced from outside. These conditions include, among others, the domain, boundary conditions and electrical or magnetic fields. They are modeled as a minimization problem and the functionals to be minimized are called *energies*. As mentioned in the introduction one common energy is the Frank–Oseen energy. In the simplest form it is the harmonic energy

$$\mathcal{J}(Q) := \int_M |\nabla Q|^2 \,\mathrm{d}V_g \;. \tag{30}$$

It would be good to know if the minimizer is orientable before it is calculated. Ball and Zarnescu [BZ11] show that a condition exists for flat 2-dimensional domains with holes. Unfortunately the transfer of this result to surfaces is beyond the scope of this thesis because it heavily depends of other results that are specific to this kind of domain.

Therefore this section will be restricted to a note on how the harmonic energy of a unit vector fields and its corresponding line field relate and an example on a torus where a line field minimizer is not orientable. This shows that the question is also relevant to the manifold setting.

Remark 6.1 (Harmonic energy). For an orientable line field the energy of the line field and the corresponding unit vector fields have a simple correlation. Let $(E_i)_i$ be a local orthonormal frame of TM with coframe ε^i. Then

$$\begin{aligned}
|\nabla P(n)|_g^2 &= \left|\nabla\left(s\left(n\otimes n-\frac{1}{m}g^{\sharp\sharp}\right)\right)\right|_g^2 \\
&= \left\langle s\nabla(n\otimes n)-\frac{1}{m}\nabla g, s\nabla(n\otimes n)-\frac{1}{m}\nabla g\right\rangle_g \\
&= s^2\left\langle \nabla_{E_i}n\otimes n\otimes\varepsilon^i+n\otimes\nabla_{E_i}n\otimes\varepsilon^i,\right. \\
&\qquad \left.\nabla_{E_i}n\otimes n\otimes\varepsilon^i+n\otimes\nabla_{E_i}n\otimes\varepsilon^i\right\rangle_g \qquad (g \text{ is parallel}) \\
&= s^2\left(\left\langle\nabla_{E_i}n,\nabla_{E_i}n\right\rangle_g\langle n,n\rangle_g\langle\varepsilon^i,\varepsilon^i\rangle_g+\left\langle\nabla_{E_i}n,n\right\rangle_g\left\langle n,\nabla_{E_i}n\right\rangle_g\langle\varepsilon^i,\varepsilon^i\rangle_g\right. \\
&\qquad \left.+\left\langle n,\nabla_{E_i}n\right\rangle_g\left\langle\nabla_{E_i}n,n\right\rangle_g\langle\varepsilon^i,\varepsilon^i\rangle_g+\langle n,n\rangle_g\left\langle\nabla_{E_i}n,\nabla_{E_i}n\right\rangle_g\langle\varepsilon^i,\varepsilon^i\rangle_g\right) \\
&= s^2\left(2\,|\nabla n|_g^2\cdot 1+0\cdot 0\cdot 1+0\cdot 0\cdot 1\right)=2s^2\,|\nabla n|_g^2\ .
\end{aligned}$$

Here we use $\langle n,n\rangle_g=1$ and

$$0=\frac{1}{2}\nabla_{E_i}(\langle n,n\rangle_g)=\frac{1}{2}\left\langle\nabla_{E_i}n,n\right\rangle_g+\frac{1}{2}\left\langle n,\nabla_{E_i}n\right\rangle_g=\left\langle\nabla_{E_i}n,n\right\rangle_g\ .$$

Therefore we define the harmonic energy as

$$\begin{aligned}
\mathcal{I}\colon \Gamma_{W^{\nabla,2}}(\mathcal{Q}^{\mathbb{S}}M)&\to\mathbb{R} \\
\mathcal{I}(Q)&:=\int_M|\nabla Q|_g^2\,\mathrm{d}V_g \\
\mathcal{J}\colon \Gamma_{W^{\nabla,2}}(\mathbb{S}M)&\to\mathbb{R} \\
\mathcal{J}(n)&:=2s^2\int_M|\nabla n|_g^2\,\mathrm{d}V_g \\
\text{such that } \mathcal{J}&=\mathcal{I}\circ P\,.
\end{aligned}$$

Before [BZ11] develops in chapter 5 the orientability criterion for minimizers on punctured flat domains, it presents an example of a domain that allows orientable and non-orientable minimizers of the harmonic energy and a boundary condition that forces the minimizer to be non-orientable. This domain is called the *stadium domain* because of its shape depicted in Figure 7. Now we want to find a similar example on the torus.

The stadium domain contains two holes to allow a boundary conditions that does not completely determine if the line field is orientable: there is no condition on the boundary of the holes and the outer boundary is a closed curve that is homotopic to the concatenation of the two generators of the fundamental group. On the outer boundary, the line field has a winding number of 1 (i. e. the auxiliary vector field $A(Q)$ has a winding number of 2) that gets split into two halfs for the non-orientable minimizer.

The fundamental group of the torus also has generators. In order to create a similar scenario, we take a line field with winding number 1 on the concatenation of the generators as shown in Figure 8.

In order to have simple calculations we use a flat metric on the torus. That is, we define the torus as the quotient $\mathbb{T}^2:=\mathbb{R}^2/\mathbb{Z}^2=\left\{[x,y]\,\middle|\,(x,y)\in\mathbb{R}^2\right\}$ with $[x,y]=\{(x+k,y+l)\,|\,k,l\in\mathbb{Z}\}$ as depicted in Figure 8. The coordinates induced by $\mathbb{R}^2$ are

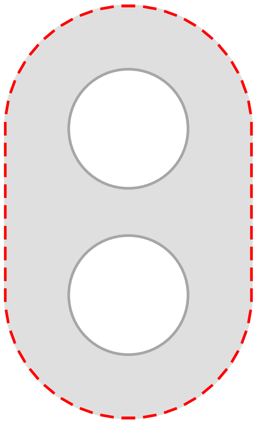
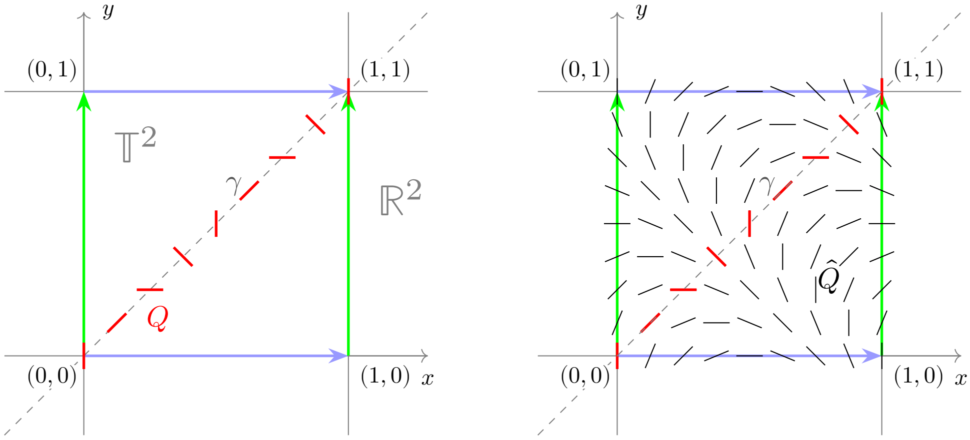


Figure 7: The stadium domain with the boundary condition that allows orientable and non-orientable minimizers of the harmonic energy. The boundary condition only predefines the line field on the outer boundary while the values on the hole boundaries are free. From [Chapter 5 BZ11].



(a) The 'boundary' condition (31) in red.

(b) The unorientable candidate for energy minimization $\hat{Q}$ on $\mathbb{T}^2$

Figure 8: The 2-dimensional torus as the quotient $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the diagonal γ. The colored edges of the square indicate how opposite sides of the unit square are identified to define the torus.

called x and y, the metric is $g = (\mathrm{d}x)^2 + (\mathrm{d}y)^2$. The covariant derivative is the usual Euclidean derivative and the area element $\mathrm{d}V_g$ is $\mathrm{d}x \otimes \mathrm{d}y$.

The torus does not have a boundary on which we can fix a condition. So we use the closed curve given by the image of $\gamma\colon [0,1] \to \mathbb{T}^2$, $\gamma(t) := [t,t]$ and set the 'boundary' condition drawn in Figure 8a.

$$Q_{\gamma(t)} := P(n_{\gamma(t)} \otimes n_{\gamma(t)}) \text{ with } n_{\gamma(t)} := \sin(2\pi t)\frac{\partial}{\partial x} + \cos(2\pi t)\frac{\partial}{\partial y} \tag{31}$$

One obvious candidate for minimizing the harmonic energy is given by translating the values on γ perpendicular to γ:

$$\begin{aligned} \hat{Q}_{[x,y]} &:= P(\hat{n}_{[x,y]}) \\ \text{with } \hat{n}_{[x,y]} &:= \sin\left(2\pi\frac{x+y}{2}\right)\frac{\partial}{\partial x} + \cos\left(2\pi\frac{x+y}{2}\right)\frac{\partial}{\partial y} =: \hat{n}^1\frac{\partial}{\partial x} + \hat{n}^2\frac{\partial}{\partial y}\ . \end{aligned}$$

Figure 8b shows $\hat{Q}$. Note that $\hat{n}$ is not well-defined since $\sin(2\pi\frac{x+1+y}{2}) = -\sin(2\pi\frac{x+y}{2})$ and analogously for y and $\hat{n}^2$ but since $P(-\hat{n}) = P(\hat{n})$, the corresponding line field $\hat{Q}$ is well-defined. We calculate the harmonic energy of $\hat{Q}$ by choosing one of the two options for $\hat{n}$. For $\mathcal{J}$ this choice is irrelevant.

$$\begin{aligned} \frac{1}{2s^2}\mathcal{J}(\hat{Q}) &= \frac{1}{2s^2}\mathcal{J}(\hat{n}) \\ &= \int_{T^2} |\nabla\hat{n}|_g^2 \,\mathrm{d}x \otimes \mathrm{d}y \\ &= \int_0^1\int_0^1 \bigg|\partial_x\hat{n}_1\frac{\partial}{\partial x}\otimes\frac{\partial}{\partial x} + \partial_x\hat{n}_2\frac{\partial}{\partial y}\otimes\frac{\partial}{\partial x} \\ &\qquad\qquad + \partial_y\hat{n}_1\frac{\partial}{\partial x}\otimes\frac{\partial}{\partial y} + \partial_y\hat{n}_2\frac{\partial}{\partial y}\otimes\frac{\partial}{\partial y}\bigg|_g^2 \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^1\int_0^1 \left(\pi\cos\left(2\pi\frac{x+y}{2}\right)\right)^2 + \left(-\pi\sin\left(2\pi\frac{x+y}{2}\right)\right)^2 \\ &\qquad\qquad + \left(\pi\cos\left(2\pi\frac{x+y}{2}\right)\right)^2 + \left(-\pi\sin\left(2\pi\frac{x+y}{2}\right)\right)^2 \mathrm{d}x\,\mathrm{d}y \\ &= \pi^2\int_0^1\int_0^1 1 + 1 \,\mathrm{d}x\,\mathrm{d}y = 2\pi^2 \end{aligned}$$

In order to find a lower limit of the energy of orientable line fields we look again at the diagonals perpendicular to γ. Let $Q \in \Gamma_{W^{\nabla,2}}(\mathcal{Q}^{\mathbb{S}}\mathbb{T}^2)$ be orientable and fulfill the 'boundary' condition. Let $n \in \Gamma_{W^{\nabla,2}}(\mathbb{S}\mathbb{T}^2)$ an orientation of Q: $P(n) = Q$. By using suitable coordinates and employing the ACL characterisation Theorem 3.10 we see that Q is continuous on almost all diagonals $D_z := \{[x,y] \mid x + y = z + k, k \in \mathbb{Z}\}$ for $z \in [0,1]$. In Figure 9 we see that a unit vector field must turn around between the intersections with γ whereas $\hat{Q}$ stays constant. Therefore we expect a higher harmonic energy for Q than for $\hat{Q}$.

To make this precise, we calculate the harmonic energy on some diagonal D_z for $z \in [0,1]$ where Q is continuous. Here $v = 2^{-\frac{1}{2}}(\frac{\partial}{\partial x} - \frac{\partial}{\partial y})$ is the direction of the diagonal. The intersections with γ are at $[\frac{z}{2}, \frac{z}{2}]$ and $[\frac{z+1}{2}, \frac{z+1}{2}]$.

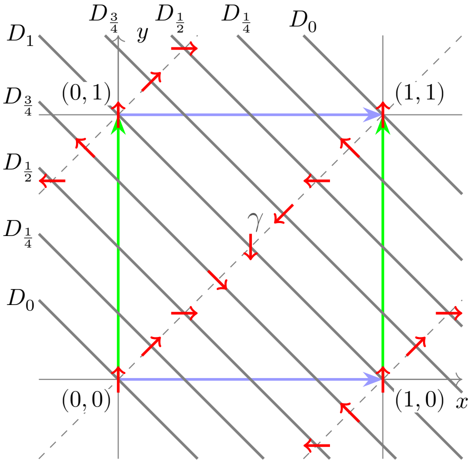


Figure 9: The diagonals D_z for $z = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ together with one of the two possible orientations for the 'boundary' condition.

$$\begin{aligned}
\int_{D_z} |\nabla n|^2 \,\mathrm{d}V_g &= \int_0^1 \left|\nabla n_{[s,z-s]}\right|^2 \sqrt{2}\,\mathrm{d}s && \text{(transformation law: } \mathrm{d}s = \mathrm{d}x - \mathrm{d}y) \\
&\geq \sqrt{2} \int_0^1 \left|\nabla_v n_{[s,z-s]}\right|^2 \mathrm{d}s \\
&= \sqrt{2} \int_0^1 \left|2^{-\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}s} n_{[s,z-s]}\right|^2 \mathrm{d}s && \text{(chain rule, } \frac{\mathrm{d}(s,z-s)}{\mathrm{d}s} = \sqrt{2}v) \\
&\geq \frac{1}{\sqrt{2}} \left(\int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}s} n_{[s,z-s]}\right| \mathrm{d}s\right)^2 && \text{(Jensen inequality)} \\
&= \frac{1}{\sqrt{2}} \left(\int_{\frac{z}{2}}^{\frac{z+1}{2}} \left|\frac{\mathrm{d}}{\mathrm{d}s} n_{[s,z-s]}\right| \mathrm{d}s + \int_{\frac{z+1}{2}}^{\frac{z}{2}+1} \left|\frac{\mathrm{d}}{\mathrm{d}s} n_{[s,2z-s]}\right| \mathrm{d}s\right)^2 \\
&= \frac{1}{\sqrt{2}} \left(\mathrm{length}(\mathbb{S}^1_{\geq 0}) + \mathrm{length}(\mathbb{S}^1_{\leq 0})\right)^2 \\
&= \frac{1}{\sqrt{2}} (\pi + \pi)^2 = \frac{4}{\sqrt{2}} \pi^2
\end{aligned}$$

$$\begin{aligned}
\text{Hence } \frac{1}{2s} \mathcal{J}(n) &= \int_{\mathbb{T}^2} |\nabla n(s)|^2 \,\mathrm{d}V_g(s) \\
&= \int_0^{\frac{\sqrt{2}}{2}} \int_{D_z} |\nabla n(s)|^2 \,\mathrm{d}s\,\mathrm{d}z \geq \frac{4\sqrt{2}}{2\sqrt{2}} \pi^2 = 2\pi^2 = \frac{1}{2s} \mathcal{J}(\hat{Q})
\end{aligned}$$

In order to have $\mathcal{J}(n) \leq \mathcal{J}(\hat{Q})$, the estimate $|\nabla n| \geq |\nabla_v n|$ must be sharp almost everywhere on almost every diagonal D_z. That means that for the perpendicular direction $v^\perp = 2^{-\frac{1}{2}}(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})$ we have $\nabla_{v^\perp} n = 0$ almost everywhere which makes n constant in this direction. This contradicts the 'boundary' condition which is not constant in the direction $v^\perp$. Therefore, the minimizer in the class of all line fields has a lower harmonic energy than the minimizer in the class of orientable line fields. Consequently, in this case, it is necessary to use the Q-tensor model and not sufficient to consider the Frank–Oseen model.

7 Outlook

The goal of thesis is to generalise the orientability results of [BZ11] to non-Euclidean domains. We do generalise the orientability criteria for Sobolev line fields on simply-connected domains and on two-dimensional domains. On the other hand we do not generalise the criteria for orientability of energy minimizers of the harmonic energy on two-dimensional domains. Therefore this can be pursued next.

While proving the orientability criterion on surfaces, we are limited to orientable surfaces because unorientable surfaces do not have a global frame. The other two main proof ingredients, namely the approximation result Theorem 5.11 and the orientability criterion for continuous line fields, hold on unorientable surfaces as well. Therefore, further work could search for a technique to transfer the orientability on loops to smooth approximations. In the case of orientable surfaces, this technique is the winding number.

Both orientability criteria are also limited by the exponent q which could not be less than 2. Indeed, Figure 2 in [BZ11] shows that for $q < 2$ there exist non-orientable Sobolev line fields on simply-connected domains. This example contains a so-called 'defect', a point where no direction can be continuously defined. The study of defects is central in a lot of liquid crystal research [Nit+18; NV12]. Hence it is a serious limitation of [BZ11] and this thesis to exclude those cases. Therefore, the study of orientability criteria for $q < 2$ provides a range of possible further research.

In order to generalise the orientability criterion on surfaces to manifolds of higher dimension, we need at least two new ideas. First the approximation theorem Theorem 5.11 is limited to two-dimensional manifolds. Secondly we need a way to transfer the orientability on loops from a Sobolev line field to the smooth approximations. Good ideas are needed to, maybe using the degree as the natural generalisation of the winding number to higher dimensions.

Declaration

I hereby confirm that the present master thesis on *Orientability in Nematic Liquid Crystal Models on Surfaces* submitted on 2022–11–03 under supervision of Prof. Dr. Oliver Sander and Dr. Hanne Hardering is solely my own work and that if any text passages or diagrams from books or other sources have been copied or used, all references have been acknowledged and fully cited. Only the quoted sources have been used.

Dresden, 2022–11–03 Signature

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